

Time Aperiodic Perturbations of Integrable Hamiltonian Systems

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February 8, 2008

Abstract

We consider a Hamiltonian $H = H^0(p) + \kappa H^1(p, q, t)$, $(p, q) \in \mathbb{R}^n \times \mathbb{T}^n$, $t \in \mathbb{R}$ where $\kappa \in \mathbb{R}$ is a small perturbation parameter and p, q are the action and angle variables respectively. The Hamiltonian generates an autonomous vector field obtained by extending the phase space making t a dependent variable and adding its conjugate variable τ . In this paper we look at a time aperiodic perturbation $H^1(p, q, t)$ which tends as $t \rightarrow \infty$ to either a time independent perturbation or a time quasiperiodic perturbation and we prove a KAM-type theorem. Extending the phase space results in the preservation under a small enough perturbation of cylinders of the extended autonomous system rather than the usual tori. To prove the theorem we transform the Hamiltonian H to a normal form which depends on fewer angles, none if possible. This transformation is done via a near identity canonical transformation. The canonical transformation is constructed using the Lie series formalism and by solving for a generating function. Because of the aperiodic time dependence, the usual Fourier series methods used to obtain the generating function no longer apply. Instead, we use Fourier transform methods to solve for the generating function and make use of an isoenergetic non-degeneracy condition which results in a shift of frequencies associated with each cylinder.

*This research was supported by ONR Grant No. N00014-97-1-0071.

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1 Introduction

Our motivation for this work is the application to the existence of “flow barriers” in two dimensional, incompressible, time-dependent fluid flows. Over the past 10 years there has been much work in applying the approach and methods of dynamical systems theory to the study of transport in fluids from the Lagrangian point of view. Suppose one is interested in the motion of a *passive tracer* in a fluid (e.g. dye, temperature, or any material that can be considered as having negligible effect on the flow), then, *neglecting molecular diffusion*, the passive tracer follows fluid particle trajectories which are solutions of

$$\dot{x} = v(x, t). \quad (1.1)$$

where $v(x, t)$ is the velocity field of the fluid flow, $x \in \mathbb{R}^n, n = 2$ or 3 . When viewed from the point of view of dynamical systems theory, the phase space of (1.1) is actually the physical space in which the fluid flow takes place. Evidently, “structures” in the phase space of (1.1) should have some influence on the transport and mixing properties of the fluid. Babiano [5] and Aref and El Naschie [1] provide recent reviews of this approach.

To make the connection with the large body of literature on dynamical systems theory more concrete let us consider a less general fluid mechanical setting. Suppose that the fluid is two-dimensional, incompressible, and inviscid. Then we know that the velocity field can be obtained from the derivatives of a scalar valued function, $\psi(x_1, x_2, t)$, known as the *streamfunction*, as follows

$$\begin{aligned} \dot{x}_1 &= \frac{\partial \psi}{\partial x_2}(x_1, x_2, t), \\ \dot{x}_2 &= -\frac{\partial \psi}{\partial x_1}(x_1, x_2, t), \end{aligned} \quad (x_1, x_2) \in \mathbb{R}^2. \quad (1.2)$$

In the context of dynamical systems theory, (1.2) is a time-dependent Hamiltonian vector field where the streamfunction plays the role of the Hamiltonian function. If the flow is time-periodic then the study of (1.2) is typically reduced to the study of a two-dimensional area preserving *Poincaré map*. Practically speaking, the reduction to a Poincaré map means that rather than viewing a particle trajectory as a curve in continuous time, one views the trajectory only at discrete intervals of time, where the interval of time is the period of the velocity field. The value of making this analogy with Hamiltonian dynamical systems lies in the fact that a variety of techniques in this area have immediate applications to, and implications for, transport and mixing processes in fluid mechanics. For example, the persistence of invariant curves in the Poincaré map (KAM curves) gives rise to barriers to transport, chaos and Smale horseshoes provide mechanisms for the “randomization” of fluid particle trajectories, an analytical technique, Melnikov’s method, allows one to estimate fluxes as well as describe the parameter regimes where chaotic fluid particle motions occur, a relatively new technique, lobe dynamics, enables one to efficiently compute transport between qualitatively different flow regimes.

Extending the constructions of Smale horseshoes, Melnikov’s method, and stable and unstable manifolds of hyperbolic trajectories to vector fields with aperiodic time-dependence has been done. However, extensions of KAM theory to vector fields with arbitrary time-dependence has not been done. This paper represents a first step in that direction.

One of the main stability results for nearly-integrable Hamiltonian systems is Kolmogorov’s theorem [2] concerning the preservation of a set of full-measure-nonresonant invariant tori. Such nearly-integrable Hamiltonian systems are generated by a real valued Hamiltonian written in action-angle variables

$$H(p, q) = H^0(p) + \kappa H^1(p, q),$$

where $p = (p_1, \dots, p_n) \in B \subset \mathbb{R}^n$, $q = (q_1, \dots, q_n) \in \mathbb{T}^n$ are, respectively, the action and angle variables, and κ is a small perturbation parameter. $\kappa H^1(p, q)$ is therefore a small perturbation of the integrable Hamiltonian $H^0(p)$. For any $p \in B$ the unperturbed angular frequencies are defined by $\tilde{\lambda}(p) = (\lambda_1, \dots, \lambda_n)(p) =$

$(\partial H^0/\partial p_1(p), \dots, \partial H^0/\partial p_n(p))$. For the integrable Hamiltonian $H^0(p)$ the equations of motion reduce to $\dot{p} = -\partial H^0/\partial q = 0$, $\dot{q} = \partial H^0/\partial p = \tilde{\lambda}(p)$ with solutions $p(t) = p_0$, $q(t) = q_0 + \tilde{\lambda}(p_0)t \pmod{2\pi}$. Consequently the phase space is foliated by invariant n -tori with quasi-periodic motions characterized by the frequency $\tilde{\lambda}(p)$. The result of adding a small perturbation to the integrable Hamiltonian $H^0(p)$ is the destruction of all tori except for those whose frequencies satisfy a non-resonance condition.

In the KAM theorem attention is restricted to tori supporting quasi-periodic motions with an appropriate nonresonant condition. Given the set of all frequencies $\Omega \subset \mathbb{R}^n$, the frequencies satisfying this nonresonant condition are called diophantine frequencies and as a subset of Ω are defined by

$$\Omega_\Gamma = \{\tilde{\lambda} \in \Omega \subset \mathbb{R}^n \mid |\tilde{\lambda} \cdot k| \geq \Gamma \|k\|^{-n}, \quad \forall k \in \mathbb{Z}^n, \quad k \neq 0\}$$

for some positive constant Γ .

The method used to prove the KAM theorem in [6] and [8], which this paper follows, is standard in classical perturbation theory. The central idea is to construct a suitable canonical transformation ψ which brings the original Hamiltonian H into a normal form which depends on fewer angles, none if possible. The transformation ψ is constructed iteratively as the composition of successive near-identity canonical transformations ϕ^1, ϕ^2, \dots . Taking the limit as the number of iterative steps goes to infinity results in the elimination of the perturbation of the Hamiltonian leaving an integrable system that is isomorphic to the original system. To construct this canonical transformation the authors in [6] and [8] make use of the Lie series method which has the advantage of avoiding any inversion and thus any reference to the implicit-function theorem. The Lie series method requires constructing a Hamiltonian called the generating function. At the k th step of the iterative process a generating function χ_k is constructed and the desired near-identity canonical transformation ϕ^k is obtained as the flow at time 1 associated with such generating function. The generating function is constructed via the nondegeneracy condition and by solving a partial differential equation.

The restriction to diophantine tori comes about from the problem of solving the partial differential equation mentioned above which has the form

$$\sum_{i=1}^n \lambda_i \frac{\partial F}{\partial q_i}(q) = G(q), \quad (1.3)$$

where the unknown function F and the known function G are defined on the torus \mathbb{T}^n and G has zero average.

In this paper we consider two similar KAM theorems where, in each, there is an aperiodic time dependent term in the perturbation. In both cases we consider an n -degree of freedom real valued nearly-integrable Hamiltonian in action-angle variables of the form

$$H(p, q, t) = H^0(p) + \kappa H^1(p, q, t),$$

where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, $q = (q_1, \dots, q_n) \in \mathbb{T}^n$, $t \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ is a small perturbation parameter. In the first case the perturbation considered has an exponentially decaying aperiodic time dependent term and a quasiperiodic term depending only on the angles q

$$H^1(p, q, t) = \sum_{k \in \mathbb{Z}^n} g_k e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} f_k(p) e_k(t) e^{ik \cdot q},$$

where $f(p)$ is a bounded function and $e_k(t)$ decays exponentially. The nature of this exponential decay is explained fully in the next section, (see definition 1). In the second case the perturbation consists of an exponentially decaying aperiodic time dependent term and a term depending in a quasiperiodic manner on the angles q and on time t

$$H^1(p, q, t) = \sum_{k \in \mathbb{Z}^{n+m}} g_k e^{ik \cdot (q, \theta t)} + \sum_{k \in \mathbb{Z}^n} f_k(p) e_k(t) e^{ik \cdot q},$$

where $f(p)$, $e_k(t)$ are as above and $\theta = (\theta_1, \dots, \theta_m)$ is the vector of basic frequencies. Although the second case seems more promising as far as applications is concern, we first prove the less technical first case. In the last section we sketch the proof of the time quasiperiodic case and state the theorem. The only difference

in the assumptions of the time quasiperiodic theorem involves the diophantine condition $\lambda \in \Omega_\Gamma$ which must hold for the larger vector $\lambda = (\tilde{\lambda}, \theta) \in \mathbb{Z}^{n+m}$ rather than just the vector $\tilde{\lambda} \in \mathbb{Z}^n$.

In both cases we consider the nearly-integrable system generated by the Hamiltonian $H(p, q, t)$ and prove a KAM type theorem where instead of tori we show the preservation of cylinders of the form $\mathbb{T}^n \times \mathbb{R}$. The proof is similar in method to the one given in [6]. First, the non-autonomous vector field is made autonomous by making time a dependent variable, $t = q_{n+1}$, and grouping it with the angles. Similarly a conjugate variable $\tau = p_{n+1}$ is grouped with the action variables. We write the Hamiltonian in a Kolmogorov-type normal form, which consists of Taylor expanding the Hamiltonian about $p = 0, \tau = 0$ and grouping terms of different orders of magnitude. Note the proof of this theorem focuses on the preservation of the cylinder with $p = 0, \tau = 0$. This can be done without loss of generality since any other cylinder can be shifted to the “zero” cylinder.

Once the Hamiltonian is in Kolmogorov normal form we seek a suitable canonical transformation which brings the Hamiltonian into a normal form which depends on fewer angles. The transformation is constructed iteratively as the product of near identity canonical transformations ϕ_k ; i.e $\phi = \phi_1 \circ \phi_2 \circ \dots \circ \phi_n$. The result is a sequence of Hamiltonians, $H_1 = H_0 \circ \phi_1, H_2 = H_1 \circ \phi_2, \dots, H_n = H_{n-1} \circ \phi_n$, which come closer to the desired normal form. Letting $n \rightarrow \infty$, we obtain H_∞ which is an integrable Hamiltonian. The sequence of canonical transformations is obtained using the Lie series method. This method and other techniques from the theory of several complex variables will require us to extend the domain of our real valued functions and consider analytic functions defined on a complex domain.

To carry out the Lie series method we introduce a generating function of the form

$$\chi(p, \tau, q, t) = \xi \cdot (q, t) + X(q, t) + \sum_{i=1}^{n+1} Y_i(q, t) p_i,$$

where $\xi \in \mathbb{R}^{n+1}$. To solve for X and Y_i we must solve a partial differential equation of the form

$$\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i}(q, t) = G(q, t), \quad (1.4)$$

where $\lambda_i = \frac{\partial H_0}{\partial p_i}(0)$ for $i = 1, \dots, n$, $\lambda_{n+1} = 1$, the given function $G(q, t)$ with zero average and the unknown function $F(q, t)$ are defined on $\mathbb{T}^n \times \mathbb{R}$. The known function $G(q, t)$ will be an expression related to the perturbation and the unknown function $F(q, t)$ will be an expression related to $X(q, t)$ or $Y_i(q, t)$.

For the time independent case with functions $F(q)$ and $G(q)$ defined on \mathbb{T}^n the partial differential equation can be solved in terms of Fourier coefficients [6]. Understandably though, the aperiodic nature of the time dependence renders the Fourier series procedure inapplicable to the problem at hand. Instead we must use Fourier transform methods to solve for $X(q, t)$ and $Y_i(q, t)$ and obtain the appropriate domains of analyticity. Consequently $F(q, t)$ is written as a Fourier series with time dependent Fourier coefficients $f_k(t)$ and these coefficients are expressed in terms of the Fourier transform of the time dependent Fourier coefficients of $G(q, t)$. Solving (1.4) for the unknown function $F(q, t)$ restricts the form of the time dependent perturbation requiring an exponential decay in time in order to obtain convergence of the Fourier coefficients $f_k(t)$.

To obtain a value for ξ and completely solve for the generating function we make use of the nondegeneracy assumption $\det(\partial \lambda_i / \partial p_i) \neq 0$ $i = 1, \dots, n$. Note this assumption implies the frequencies are functionally independent and since $\lambda_{n+1} = 1$, the ratios of the frequencies λ_i $i = 1, \dots, n$ to λ_{n+1} are functionally independent as well. Thus we obtain what is called an isoenergetic nondegeneracy condition. This condition is the key feature in solving for ξ .

After applying the iterative lemma n times and taking $n \rightarrow \infty$ we obtain an integrable Hamiltonian $H_\infty(\tilde{p}, \tilde{\tau}, \tilde{q}, \tilde{t})$ in the transformed variables $(\tilde{p}, \tilde{\tau}, \tilde{q}, \tilde{t})$ which generates an autonomous vector field in $\mathbb{T}^n \times \mathbb{R}$ with solutions of the form $(\tilde{p}, \tilde{\tau}) = (p_0, \tau_0)$, $\tilde{q} = (1 + \kappa E \zeta_A) \tilde{\lambda} t + q_0$, $\tilde{t} = (1 + \kappa E \zeta_A) t$. These solutions imply the phase space $\mathbb{R}^{n+1} \times \mathbb{T}^n \times \mathbb{R}$ is foliated by invariant infinite cylinders each sustaining winding motions identified by the frequency $\tilde{\lambda}(p_0)$ and evolving in the \tilde{t} direction.

In Appendix G we present the Rossby wave flow [13] with a time decaying perturbation and show this system satisfies the hypothesis of the theorem presented in this paper.

2 Preliminaries

Although we begin by considering a real Hamiltonian dynamical system, techniques from several complex variables will be use in the analysis. This requires complex extensions of the original functions defined in \mathbb{R}^m to functions defined in \mathbb{C}^m . We begin describing these complex extensions with some notation. Denote the open ball of radius ρ centered at p_0 by B so that $p_0 \in B \subset \mathbb{R}^m$, and we assume without loss of generality $\rho < 1$. Also, denote $q \equiv (q_1, \dots, q_m) \in \mathbb{T}^m$ where we identify functions on \mathbb{T}^m with functions on \mathbb{R}^m that are 2π periodic in each q_1, \dots, q_m . The complex extension of $B \times \mathbb{T}^m$ is given by

$$D_{\rho, p_0} = \{(p, q) \in \mathbb{C}^{2m} \mid \|p - p_0\| \leq \rho, \operatorname{Re} q \in \mathbb{R}^n \bmod 2\pi, \|\operatorname{Im} q\| \leq \rho\},$$

where $\operatorname{Im} q = (\operatorname{Im} q_1, \dots, \operatorname{Im} q_m)$ and $\|v\| = \max_i |v_i|$ for $v \in \mathbb{C}^m$. Define \mathcal{A}_{ρ, p_0} as the set of all complex continuous functions defined on D_{ρ, p_0} , analytic in the interior of D_{ρ, p_0} , 2π periodic in the variables q_1, \dots, q_m , and real for real values of the variables.

Consider the following non-autonomous Hamiltonian

$$H(p, q, t) = H^0(p) + \kappa H^1(p, q, t),$$

where $p \in \mathbb{R}^n, q \in \mathbb{T}^n$ are the action and angle variables respectively, $t \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ is a small perturbation parameter. The nature of the time dependence is explained at the end of this section. Hamilton's equations are given by

$$\dot{p} = -\frac{\partial H}{\partial q} = -\kappa \frac{\partial H^1}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H^0}{\partial p} + \kappa \frac{\partial H^1}{\partial p}. \quad (2.1)$$

For the system generated by the integrable Hamiltonian $H^0(p)$ we define the angular frequencies $\tilde{\lambda}(p) = (\frac{\partial H^0}{\partial p_1}, \dots, \frac{\partial H^0}{\partial p_n}) = (\lambda_1, \dots, \lambda_n)$ and Hamilton's equations reduce to $\dot{p} = -\partial H^0 / \partial q = 0$, $\dot{q} = \partial H^0 / \partial p = \tilde{\lambda}$ with solutions $p(t) = p_0$, $q(t) = q_0 + \tilde{\lambda}(p_0)t$. Consequently the phase space is foliated by invariant n-tori, each of which sustains quasi-periodic motions characterized by the frequency $\tilde{\lambda}(p_0)$.

To analyze the full Hamiltonian $H(p, q, t)$ with the time dependent perturbation $H^1(p, q, t)$ we change this non-autonomous system into an autonomous system by making time a dependent variable and effectively extending the phase space from $\mathbb{R}^n \times \mathbb{T}^n$ to $\mathbb{R}^n \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}$. For notation sake we group t with the angles q and its conjugate variable τ with the actions p and write the Hamiltonian $H(p, \tau, q, t)$ with $(p, \tau, q, t) \equiv (p', q') \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}$

$$H(p', q') \equiv H(p, \tau, q, t) = H^0(p) + \kappa H^1(p, q, t) + \tau$$

or

$$H(p', q') = \tilde{H}^0(p, \tau) + H^1(p, q, t) = \tilde{H}^0(p') + \kappa H^1(p', q'), \quad (2.2)$$

where $\tilde{H}^0(p') = H^0(p) + \tau$. Given $0 < \sigma < \rho < 1$, we define the complex extension of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}$

$$D_{\rho, \sigma, p_0} = \{(p, \tau, q, t) \in \mathbb{C}^{2n+2} \mid \|p - p_0\| \leq \rho, |\tau| \leq \sigma, \operatorname{Re} q \in \mathbb{R}^n \bmod 2\pi, \|\operatorname{Im} q\| \leq \rho, |\operatorname{Im} t| \leq \sigma\},$$

and define $\mathcal{A}_{\rho, \sigma, p_0}$ as the set of all complex continuous functions defined on D_{ρ, σ, p_0} , analytic in the interior of D_{ρ, σ, p_0} , 2π periodic in the variables q_1, \dots, q_n , and real for real values of the variables. Hamilton's equations for the new autonomous Hamiltonian are given by

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -\kappa \frac{\partial H^1}{\partial q}, \\ \dot{\tau} &= -\frac{\partial H}{\partial t} = -\kappa \frac{\partial H^1}{\partial t}, \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{\partial H^0}{\partial p} + \kappa \frac{\partial H^1}{\partial p}, \\ \dot{t} &= \frac{\partial H}{\partial \tau} = 1. \end{aligned} \quad (2.3)$$

Of the new Hamilton's equations, (2.3), the first, third and fourth equations are equivalent to the original system (2.1). For the system generated by the Hamiltonian $\tilde{H}^0(p')$ Hamilton's equations reduce to $\dot{p}' = -\partial\tilde{H}^0/\partial q' = 0$, $\dot{q}' = \partial\tilde{H}^0/\partial p' = \lambda(p')$ with solutions $p'(t) = p'_0$, $q'(t) = \lambda(p'_0)t + q_0$ where

$$\lambda(p') = \frac{\partial\tilde{H}^0}{\partial p'}(p') = \left(\frac{\partial\tilde{H}^0}{\partial p_1}, \dots, \frac{\partial\tilde{H}^0}{\partial p_n}, \frac{\partial\tilde{H}^0}{\partial \tau}\right) = (\lambda_1, \dots, \lambda_n, 1).$$

We will often write $\lambda = (\tilde{\lambda}, 1) = (\lambda_1, \dots, \lambda_n, 1)$. It is instructive to write out the solution $q'(t) = (q_1(t), \dots, q_n(t), t) = (\lambda_1(p'), \dots, \lambda_n(p'), 1)t + q_0$ where $q_0 \in \mathbb{T}^n \times \mathbb{R}$ and the $n+1$ term of q_0 is zero. Consequently the phase space $(p', q') \in (\mathbb{R}^{n+1}, \mathbb{T}^n \times \mathbb{R})$ is foliated by invariant infinite cylinders each sustaining winding motions characterized by the frequency $\tilde{\lambda}(p_0)$ which evolve along the t direction.

For a vector $v \in \mathbb{C}^n$ we define the norm $|v| = \sum_i^n |v_i|$. For a scalar valued function $f \in \mathcal{A}_{\rho, \sigma, p_0}$ we use the norm, $\|f\|_{\rho, \sigma, p_0} \equiv \sup_{(p', q') \in D_{\rho, \sigma, p_0}} |f(p', q')|$. For vector valued functions $f = (f_1, \dots, f_{2n+2})$ with values in \mathbb{C}^{2n+2} we define $f \in \mathcal{A}_{\rho, \sigma, p_0}$ if $f_i \in \mathcal{A}_{\rho, \sigma, p_0}$, ($i = 1, \dots, 2n+2$), and the norm is defined as follows

$$\|f\|_{\rho, \sigma, p_0} = \max_i \|f_i\|_{\rho, \sigma, p_0}.$$

For a $(2n+2) \times (2n+2)$ matrix C whose entries $C_{i,j}$ belong to $\mathcal{A}_{\rho, \sigma, p_0}$ we write the norm of C as $\|C\|_{\rho, \sigma, p_0} = \max_{i,j} \|C_{i,j}\|_{\rho, \sigma, p_0}$. If the matrix C is a constant matrix then $\|C\| = \max_{i,j} |C_{i,j}|$.

For the integrable Hamiltonian H^0 recall the frequency map

$$\tilde{\lambda} : p \rightarrow \frac{\partial H^0}{\partial p}(p) \equiv \tilde{\lambda}(p).$$

The set of all possible frequencies in \mathbb{R}^n is denoted by $\Omega \equiv \tilde{\lambda}(B)$. The subset of diophantine frequencies which will be important throughout the paper is defined for some positive constant Γ by

$$\Omega_\Gamma = \{\tilde{\lambda} \in \Omega \subset \mathbb{R}^n \mid |\tilde{\lambda} \cdot k| \geq \Gamma \|k\|^{-n}, \quad \forall k \in \mathbb{Z}^n, k \neq 0\}.$$

Given a function $f(p, \tau, q, t) = f(p', q') \in \mathcal{A}_{\rho, \sigma}$ 2π periodic in each of the q_i 's except $q_{n+1} = t$, $\overline{f}(p^*)$ will denote the average of the function f over the angles q_1, \dots, q_n and evaluated at p^*

$$\overline{f}(p^*, t) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} f(p^*, q, t) dq_1 \dots dq_n.$$

Given a function $f(p, \tau, q, t) = f(p', q') \in \mathcal{A}_{\rho, \sigma}$ 2π periodic in each of the q_i 's except $q_{n+1} = t$, we will denote the average of the function $f(p', q')$ over the angles q_1, \dots, q_n and over t evaluated at p^* as follows

$$\overline{\overline{f}}(p^*) = \lim_{T \rightarrow \infty} \frac{1}{2T} \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \dots \int_0^{2\pi} \int_{-T}^T f(p^*, q, t) dq_1 \dots dq_n dt. \quad (2.4)$$

Note for a given Hamiltonian $H \in \mathcal{A}_{\rho, \sigma, p_0}$, one can assume $\|H\|_{\rho, \sigma} < 1$. This can be obtained by the change of variables $(p', q') \rightarrow (\alpha p', q')$ with a suitable positive constant α .

Now we address the topic of the time dependence of the perturbation $H^1(p, q, t)$. Since we want to consider a time aperiodic dependence and the perturbation plays a key role in the solution of the canonical transformation, the perturbation has to have a particular form for certain Fourier transforms to converge. The form of the perturbation consists of an exponential decay in time of the time dependent part of the perturbation. The exponential decay in time of the perturbation is described by the following definition.

Definition 2.1

We say a complex valued function $f(z)$, $z = x + iy$ with $x, y \in \mathbb{R}$, is of $(C_1, C_2, c_1, c_2, \nu, \mu)$ -exponential order with respect to x and write

$$f(z) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)x}) & (x \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)x}) & (x \rightarrow -\infty) \end{cases},$$

where $\nu, \mu, \varepsilon > 0, \nu > \varepsilon, \mu > \varepsilon$ if there exist constants $C_1, C_2, c_1 > 0, c_2 < 0$ such that

$$|f(z)| \leq C_1 e^{-(\nu-\varepsilon)x} \quad \text{for } 0 < c_1 < x < \infty \text{ and}$$

$$|f(z)| \leq C_2 e^{(\mu-\varepsilon)x} \quad \text{for } -\infty < x < c_2 < 0.$$

The iterative lemma will require the perturbation to have a particular form. This form is defined below.

Definition 2.2

We will refer to a function $F(p', q')$ as having $(C_1, C_2, c_1, c_2, \nu, \mu)p', q'$ -exponential form if it can be expressed as follows

$$F(p', q') = f(p') + r(p', q') + g(p', q'),$$

$$r(p', q') = \sum_{k \in \mathbb{Z}^n} s_k(p') e^{ik \cdot q},$$

$$g(p', q') = \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

where $F(p', q') \in \mathcal{A}_{\rho, \sigma}$ and $e_k(t), t = t_R + it_I, t_R, t_I \in \mathbb{R}$, is of C_1, C_2, c_1, c_2 -exponential order with respect to t_R . When it is not important for context to specify the constants $C_1, C_2, c_1, c_2, \nu, \mu$ we will simply refer to functions of exponential order with respect to x or functions of p', q' -exponential form.

3 Statement of the Theorem

We now state the main theorem in terms of the Hamiltonian $H(p', q') = H^0(p) + \kappa H^1(p, q, t) + \tau$.

Theorem 3.1

Consider the Hamiltonian $H(p', q') = H^0(p) + \kappa H^1(p, q, t) + \tau = \tilde{H}^0(p') + \kappa H^1(p', q')$ of $(C_1, C_2, c_1, c_2, \nu, \mu)p', q'$ -exponential form defined on $B \times \mathbb{T}^n$ where $\tilde{H}^0(p') = H^0(p) + \tau$. Fix $p'_0 \in B$ and denote

$$\lambda = \lambda(p'_0) = \frac{\partial \tilde{H}^0}{\partial p'}(p'_0), \quad \tilde{C}_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p_i \partial p_j}(p'_0), \quad C_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(p'_0), \quad \mathbb{I} = \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix}.$$

Assume there exists positive numbers $\Gamma, \gamma, \rho, \sigma, d, \kappa$, all less than one and L , such that $\rho > \sigma$ and

$$\tilde{\lambda} \in \Omega_\Gamma, \quad |\tilde{\lambda}| < L,$$

$$H^0, H^1 \in \mathcal{A}_{\rho, \sigma, p_0},$$

$$d \|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\| \leq d^{-1} \|\tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n,$$

moreover, $\|H\|_{\rho, \sigma, p_0} < 1$, (the last condition will imply the isoenergetic nondegeneracy condition on \mathbb{I}).

Then there exists positive numbers $E, \kappa', E', \zeta', f, c_3, \gamma, \rho'$ and σ' , such that if $\|H^1\|_{\rho, \sigma, p_0} \leq E$ one can construct a canonical analytic change of variables

$$\psi : D_{\rho', \sigma'} \rightarrow D_{\rho, \sigma, p_0},$$

$$(\mathcal{P}', \mathcal{Q}') \mapsto \psi(\mathcal{P}', \mathcal{Q}') = (p', q'),$$

with $\psi \in \mathcal{A}_{\rho', \sigma'}$, which brings the Hamiltonian H into the form $H' = H \circ \psi$ given by

$$H'(\mathcal{P}', \mathcal{Q}') = (H \circ \psi)(\mathcal{P}', \mathcal{Q}') = a' + (1 + \kappa' E' \zeta') \lambda \cdot \mathcal{P}' + \mathcal{O}(\|\mathcal{P}'\|^2), \quad (3.1)$$

where $a' \in \mathbb{R}$. The change of variables is near identity in the sense that

$$\|\psi - \text{identity}\|_{\rho', \sigma} \rightarrow 0 \text{ as } \|H^1\|_{\rho, \sigma, p_0} \rightarrow 0.$$

In particular, we can take

$$\begin{aligned} \kappa E &= \left(\frac{\sigma}{32}\right)^{2\aleph} \frac{d^4 f^4 \sigma'}{2^{12} \Lambda^2}, \quad \aleph = 10n + 9, \\ \Lambda &= \frac{3^n 2^{2n+12} \varpi^3 (6n+6)^4 (c_3+1)^2}{\rho' \Gamma^3 \gamma^2} e^{\nu+\mu}, \quad \varpi = 2^{4n+1} \left(\frac{n+1}{e}\right)^{n+1}. \end{aligned}$$

In the new coordinates Hamilton's Equations are given by

$$\dot{\mathcal{Q}}' = \frac{\partial H'}{\partial \mathcal{P}'}(\mathcal{P}', \mathcal{Q}') = (1 + \kappa' E' \zeta') \lambda + \mathcal{O}(\|\mathcal{P}'\|),$$

$$\dot{\mathcal{P}}' = -\frac{\partial H'}{\partial \mathcal{Q}'}(\mathcal{P}', \mathcal{Q}') = \mathcal{O}(\|\mathcal{P}'\|^2).$$

The solutions are given by

$$\mathcal{Q}'(t) = (1 + \kappa' E' \zeta') \lambda t + \mathcal{Q}'_0 \quad \mathcal{P}'(t) = \mathcal{P}'_0 = (\mathcal{P}_0, \tilde{\tau}_0) = 0.$$

We can write out explicitly these solutions $\mathcal{Q}'(t) = (\mathcal{Q}_1(t), \dots, \mathcal{Q}_n(t), \mathcal{T}(t)) = (1 + \kappa' E' \zeta') \lambda t + \mathcal{Q}'_0 = (1 + \kappa' E' \zeta')(\tilde{\lambda}, 1)t + \mathcal{Q}'_0$ which indicates in the new coordinate system the phase space $\mathbb{R}^n \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}$ is foliated by invariant infinite cylinders each sustaining quasi-periodic motions identified by the frequency $\tilde{\lambda}(\mathcal{P}_0)$ and evolving in the t direction. Note under the transformation the time variable $\mathcal{T}(t) = (1 + \kappa' E' \zeta')t$ has shifted by a constant proportional to the size of the perturbation.

4 The Normal Form

We begin the proof with a trivial rearrangement of the Hamiltonian that consists of Taylor expanding in p, τ around $p' = 0$. Recall the Hamiltonian

$$H(p', q') = H^0(p) + \tau + \kappa H^1(p, q, t) = \tilde{H}^0(p') + \kappa H^1(p', q').$$

Taylor expanding the Hamiltonian $H(p', q')$ about $p' = 0$ we obtain

$$\begin{aligned} H(p', q') &= H(0, q') + \frac{\partial H}{\partial p'}(0, q') \cdot p' + \frac{1}{2} \frac{\partial^2 H}{\partial p'^2}(0, q')(p', p') + \mathcal{O}(\|p'\|^3) \\ &= \tilde{H}^0(0) + \kappa H^1(0, q') + \lambda \cdot p' + \kappa \partial_{p'} H^1(0, q') \cdot p' \\ &\quad + \frac{1}{2} \partial_{p'}^2 \tilde{H}^0(0)(p', p') + \kappa \frac{1}{2} \partial_{p'}^2 H^1(0, q')(p', p') + \mathcal{O}(\|p'\|^3). \end{aligned}$$

The terms can be grouped by order of magnitude in p'

$$H(p', q') = a + \lambda \cdot p' + A(q') + B(q') \cdot p' + \frac{1}{2} \sum_{i,j}^{n+1} C_{i,j}(q') p'_i p'_j + R(p', q'), \quad (4.1)$$

where

$$\begin{aligned} a &= \overline{\overline{H(0)}} = \overline{\tilde{H}^0(0)} + \kappa \overline{\overline{H^1(0)}} = \tilde{H}^0(0) + \kappa \overline{\overline{H^1(0)}}, \\ \kappa A(q') &= H(0, q') - a = \kappa(H^1(0, q') - \overline{\overline{H^1(0)}}), \end{aligned}$$

$$\begin{aligned}\kappa B_i(q') &= \frac{\partial H}{\partial p'_i}(0, q') - \lambda_i = \kappa \frac{\partial H^1}{\partial p'_i}(0, q'), \\ C_{ij}(q') &= \frac{\partial^2 H}{\partial p'_i \partial p'_j}(0, q') = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(0) + \kappa \frac{\partial^2 H^1}{\partial p'_i \partial p'_j}(0, q'),\end{aligned}$$

and $a \in \mathbb{R}$, $A, B_i, C_{i,j}, R \in A_{\rho, \sigma}$, $R = \mathcal{O}(\|p'^3\|)$. This Hamiltonian has the desired form (3.1) of Theorem 3.1 except for the terms $A(q')$ and $B(q')$ which constitute the actual perturbation of the Hamiltonian (4.1).

5 Bounds on the Normal Form in terms of the Original Hamiltonian

In what follows we present several results that allow us to reformulate Theorem 3.1 for the Hamiltonian (4.1) in normal form. Since we are interested in estimates for $A, B_i, C_{i,j}$ and these are defined in terms of derivatives of the original Hamiltonian, we often make use of Cauchy's inequality. For scalar valued functions $f \in \mathcal{A}_{\rho, \sigma}$, $\delta < \sigma < \rho$ and nonnegative integers $k_i, l_i, i = 1, \dots, n+1$

$$\left| \frac{\partial^{k_1 + \dots + k_{n+1} + l_1 + \dots + l_{n+1}}}{\partial p_1^{k_1} \dots \partial p_{n+1}^{k_{n+1}} \partial q_1^{l_1} \dots \partial q_{n+1}^{l_{n+1}}} f(p, q) \right| \leq \frac{k_1! \dots k_{n+1}! l_1! \dots l_{n+1}!}{\delta^{k_1 + \dots + k_{n+1} + l_1 + \dots + l_{n+1}}} \|f\|_{\rho, \sigma}, \quad \forall (p, q) \in \mathcal{D}_{\rho - \delta, \sigma - \delta}.$$

Lemma 5.1

Recall the Hamiltonian of interest $H(p', q') = H^0(p) + \kappa H^1(p, q, t) + \tau = \tilde{H}^0(p') + \kappa \tilde{H}^1(p', q')$ where $\tilde{H}^0(p') = H^0(p) + \tau$. Given the $n \times n$ matrix

$$\tilde{C}_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p_i \partial p_j}(0),$$

assume there is a positive number $d < 1$ such that $d\|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\| \leq d^{-1}\|\tilde{v}\| \quad \forall \tilde{v} \in \mathbb{C}^n$. Define the $(n+1) \times (n+1)$ matrix

$$C_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(0),$$

it follows $\|C^* v\| \leq d^{-1}\|v\| \quad \forall v \in \mathbb{C}^{n+1}$. Proof See appendix B.

Lemma 5.2 (Lemma on bounds)

Given a constant E such that $\|H^1\|_{\rho, \sigma} \leq E$

1. $\max(\|A\|_{\rho, \sigma}, \|B\|_{\rho, \sigma}) < 2\frac{E}{\sigma} = E_1$.

2. There exists a positive number $m \leq 1$ such that $\|Cv\|_{\rho, \sigma} \leq m^{-1}\|v\|, \quad \forall v \in \mathbb{C}^{n+1}$. In particular we take $m = \frac{d}{2}$. Proof See appendix B.

Lemma 5.3 (Isoenergetic Nondegeneracy Condition)

Assume $\sum |\lambda'_i| < L$. Given $\|\tilde{C}^* \tilde{v}\|_{\rho, \sigma} \geq d\|\tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n, 2m > L$. It follows $\|\mathbb{I}v\|_{\rho, \sigma} \geq l\|v\|, \quad \forall v \in \mathbb{C}^{n+2}$ for some $l = \frac{1}{2}|d - L|$. Proof See appendix B.

Lemma 5.4

Given $\|\mathbb{I}v\|_{\rho,\sigma} \geq l\|v\|_{\rho,\sigma}$ with $l = \frac{1}{2}|d - L|$. It follows

$$\left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|_{\rho,\sigma} \geq f\|v\|_{\rho,\sigma},$$

where $f = |l - \frac{\kappa E_1}{\sigma^2}|$.

Proof See appendix B.

6 Re-statement of the Theorem for the Normal Form

One can deduce, based on the bounds on the normal form in terms of the original Hamiltonian, theorem 3.1 from the following theorem.

Theorem 6.1 *Main Theorem*

For given positive numbers $\kappa, \Gamma, \gamma, \rho, \sigma, m$ all less than one and L consider the Hamiltonian $H(p', q') = U(p', q') + P(p', q')$ of $(C_1, C_2, c_1, c_2, \nu, \mu)p', q'$ -exponential form defined in $D_{\rho,\sigma}$ by

$$U(p', q') = a + \lambda \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j} p'_i p'_j + R(p', q'),$$

$$P(p', q') = \kappa \left(A(q') + \sum_i B_i(q') p'_i \right),$$

with $\|H\|_{\rho,\sigma} < 1$, where $\tilde{\lambda} \in \Omega_\Gamma$ and $|\tilde{\lambda}| < L, A, B_i, C_{i,j}, R \in \mathcal{A}_{\rho,\sigma}$ and R is of order $\|p'\|^3$. Assume

$$2m\|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n,$$

$$\|Cv\|_{\rho,\sigma} < m^{-1}\|v\|, \quad \forall v \in \mathbb{C}^{n+1}.$$

Then there exists positive numbers $E_1, \kappa_\infty, E_1^\infty, \zeta_\infty, \rho', \sigma', m', f$ all less than one and L', c_3, γ also positive with

$$\rho' < \rho, \sigma' < \sigma, m' < m,$$

$$f\|v\| \leq \left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

such that if

$$\max(\|A\|_{\rho,\sigma}, \|B\|_{\rho,\sigma}) < E_1,$$

we can construct a canonical analytic change of variables

$$\phi : D_{\rho',\sigma'} \rightarrow D_{\rho,\sigma},$$

with $\phi \in \mathcal{A}_{\rho',\sigma'}$, which brings the Hamiltonian H into the form

$$H'(\mathcal{P}, \mathcal{Q}) = (H \circ \phi)(\mathcal{P}, \mathcal{Q}) = a' + \lambda' \cdot \mathcal{P} + R(\mathcal{P}, \mathcal{Q}),$$

where H' is of p', q' -exponential form, $\lambda' = (1 + \kappa_\infty E_1^\infty \zeta_\infty) \lambda$, $a' \in \mathbb{R}$ and $R = \mathcal{O}(\|\mathcal{P}\|^2) \in \mathcal{A}_{\rho',\sigma'}$.

The change of variables is near the identity in the sense that

$$\|\phi - \text{identity}\|_{\rho',\sigma'} \rightarrow 0 \quad \text{as} \quad \|P\|_{\rho,\sigma} \rightarrow 0.$$

In particular, we can take

$$\kappa E_1 = \left(\frac{\sigma}{32}\right)^{2\aleph} \frac{m^4 f^4 \sigma'}{2^8 \Lambda^2}, \quad \aleph = 10n + 9,$$

$$\Lambda = \frac{3^n 2^{2n+12} \varpi^3 (6n+6)^4 (c_3+1)^2}{\rho' \Gamma^3 \gamma^2} e^{\nu+\mu}, \quad \varpi = 2^{4n+1} \left(\frac{n+1}{e}\right)^{n+1}.$$

In the new coordinates Hamilton's Equations are given by

$$\dot{\mathcal{Q}}' = \frac{\partial H'}{\partial \mathcal{P}'}(\mathcal{P}', \mathcal{Q}') = (1 + \kappa_\infty E_1^\infty \zeta_\infty) \lambda + \mathcal{O}(\|\mathcal{P}'\|),$$

$$\dot{\mathcal{P}}' = -\frac{\partial H'}{\partial \mathcal{Q}'}(\mathcal{P}', \mathcal{Q}') = \mathcal{O}(\|\mathcal{P}'\|^2).$$

The solutions are given by

$$\mathcal{Q}'(t) = (1 + \kappa_\infty E_1^\infty \zeta_\infty) \lambda t + \mathcal{Q}'_0 \quad \mathcal{P}'(t) = \mathcal{P}'_0 = (\mathcal{P}_0, \tilde{\tau}_0) = 0.$$

We can write out explicitly these solutions $\mathcal{Q}'(t) = (\mathcal{Q}_1(t), \dots, \mathcal{Q}_n(t), \mathcal{T}(t)) = (1 + \kappa_\infty E_1^\infty \zeta_\infty) \lambda t + \mathcal{Q}'_0 = (1 + \kappa_\infty E_1^\infty \zeta_\infty)(\tilde{\lambda}, 1)t + \mathcal{Q}'_0$ which indicates in the new coordinate system the phase space $\mathbb{R}^n \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}$ is foliated by invariant infinite cylinders each sustaining quasi-periodic motions identified by the frequency $\tilde{\lambda}(\mathcal{P}_0)$ and evolving in the t direction. Note under the transformation, the time variable $\mathcal{T}(t) = (1 + \kappa_\infty E_1^\infty \zeta_\infty)t$ has shifted by a constant proportional to the size of the perturbation.

7 Perturbations tending to q -Periodicity and t Quasiperiodicity

We are considering two similar KAM theorems each involving an n -degree of freedom real valued nearly-integrable Hamiltonian in action-angle variables of the form

$$H(p, q, t) = H^0(p) + \kappa H^1(p, q, t),$$

where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$, $q = (q_1, \dots, q_n) \in \mathbb{T}^n$, $t \in \mathbb{R}$ and κ is small. After extending H to the complexified domain $D_{\rho, \sigma}$ we have $H \in \mathcal{A}_{\rho, \sigma}$. In the first case the perturbation considered has an exponentially decaying aperiodic time dependent term and a quasiperiodic term depending only on the angles q

$$H^1(p, q, t) = \sum_{k \in \mathbb{Z}^n} g_k e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} f_k(p) e_k(t) e^{ik \cdot q},$$

where $f_k(p)$ is a bounded function and $e_k(t)$, $t = t_R + it_I$, is of exponential order with respect to t_R . In the second case the perturbation consists of an exponentially decaying aperiodic time dependent term and a term depending in a quasiperiodic manner on the angles q and on time t

$$H^1(p, q, t) = \sum_{k \in \mathbb{Z}^{n+1}} g_k e^{ik \cdot (q, \theta t)} + \sum_{k \in \mathbb{Z}^n} f_k(p) e_k(t) e^{ik \cdot q},$$

where $f_k(p)$, $e_k(t)$ are as above and $\theta = (\theta_1, \dots, \theta_m)$ is the vector of basic frequencies.

For each of the cases above, it is necessary for the application of the Lie series method to construct a generating function. For this purpose we will formulate an expression in terms of the perturbation $H^1(p, q, t)$ evaluated at $p = 0$, call it $G(q, t)$, and solve the partial differential equation

$$\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i}(q, t) = G(q, t). \quad (7.1)$$

For the first case, the given function has the form

$$G(q, t) = G_A(q, t) + G_Q(q) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^n} \tilde{g}_l e^{il \cdot q},$$

and we look for a solution of the form

$$F(q, t) = F_A(q, t) + F_Q(q) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^n} \tilde{f}_l e^{il \cdot q}.$$

Substituting in (7.1) we obtain

$$\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i} = \sum_{k \in \mathbb{Z}^n} \left[i\tilde{\lambda} \cdot k f_k(t) + \frac{df_k(t)}{dt} \right] e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^n} i(\tilde{\lambda} \cdot l) \tilde{f}_l e^{il \cdot q}.$$

Clearly $\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i} - G = 0$ becomes

$$\sum_{k \in \mathbb{Z}^n} \left[i(\tilde{\lambda} \cdot k) f_k(t) + \frac{df_k(t)}{dt} - g_k(t) \right] e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^n} \left[i(\tilde{\lambda} \cdot l) \tilde{f}_l - \tilde{g}_l \right] e^{il \cdot q} = 0, \quad (7.2)$$

(7.2) is satisfied if

$$\begin{cases} i(\tilde{\lambda} \cdot k) f_k(t) + \frac{df_k(t)}{dt} = g_k(t) \\ \tilde{f}_l = \frac{\tilde{g}_l}{i(\tilde{\lambda} \cdot l)} \end{cases}$$

and the problem reduces to two problems. Similarly for the second case, the given function has the form

$$G(q, t) = G_A(q, t) + G_Q(q, t) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^{n+m}} \tilde{g}_l e^{il \cdot (q, \theta t)},$$

we look for a function of the form

$$F(q, t) = F_A(q, t) + F_Q(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q} + \sum_{l \in \mathbb{Z}^{n+m}} \tilde{f}_l e^{il \cdot (q, \theta t)},$$

and the problems to be solved are

$$\begin{cases} i(\tilde{\lambda} \cdot k) f_k(t) + \frac{df_k(t)}{dt} = g_k(t) \\ \tilde{f}_l = \frac{\tilde{g}_l}{i(\tilde{\lambda}, \theta) \cdot l} \end{cases}.$$

Therefore in both KAM theorems solving (7.1) is equivalent to solving two p.d.e's; one p.d.e involving quasiperiodic dependent functions, $(G_Q(q), F_Q(q))$ or $(G_Q(q, t), F_Q(q, t))$ and a p.d.e involving p', q' -exponential form functions $(G_A(q, t), F_A(q, t))$. In the next section we give the result for solving the p.d.e involving quasiperiodic functions.

8 P.D.E on a Torus

Consider the real valued functions $F(q)$ and $G(q)$ where $q = (q_1, \dots, q_n) \in \mathbb{C}^n$ and $\text{Re } q = (\text{Re } q_1, \dots, \text{Re } q_n) \in \mathbb{R}^n \bmod 2\pi$. That is, $F(q)$ and $G(q)$ are functions on \mathbb{C}^n , 2π periodic with respect to the real part of each q_1, \dots, q_n . Although we are concerned with real Hamiltonian systems, we will use techniques from several complex variables theory in the analysis. This will require complex extensions of functions originally defined on \mathbb{R}^n to \mathbb{C}^n . Fix $\rho < 1$, we define the complex extension of \mathbb{T}^n

$$D_\rho = \{q \in \mathbb{C}^n \mid \text{Re } q \in \mathbb{R}^n \bmod 2\pi, \quad \|\text{Im } q\| \leq \rho\},$$

where $\text{Re } q \equiv (\text{Re } q_1, \dots, \text{Re } q_n)$, $\text{Im } q \equiv (\text{Im } q_1, \dots, \text{Im } q_n)$. We define \mathcal{A}_ρ , the set of all complex, continuous functions defined on D_ρ that are analytic in the interior of D_ρ and real for real values of the variables.

The following lemma gives a useful bound for the Fourier coefficients of a function $G(q)$ with a 2π periodic dependence on q .

Lemma 8.1

Assume $G \in \mathcal{A}_\rho$ and $G(q) = \sum_{k \in \mathbb{Z}^n} g_k e^{ik \cdot q}$, then for every $k \in \mathbb{Z}^n$ $|g_k| \leq \|G\|_\rho e^{-|k|\rho}$, where $|k| = \sum_i |k_i|$. Proof see appendix D.

The following lemma gives the existence and uniqueness of solutions of a p.d.e involving quasi-periodic functions. The lemma also gives analyticity results for the solution, $F(q)$, of the p.d.e given analyticity restrictions on the known function $G(q)$.

Lemma 8.2 Consider the following linear partial differential equation

$$\sum_{i=1}^n \lambda_i \frac{\partial F}{\partial q_i}(q) = G(q), \quad (8.1)$$

where F and G are functions defined on the torus \mathbb{T}^n , and assume $\lambda = (\lambda_1, \dots, \lambda_n) \in \Omega_\Gamma$, for some $\Gamma > 0$, and $G \in \mathcal{A}_\rho$ for some positive $\rho < 1$ with $\overline{G} = 0$. Then, for some positive $\delta < \rho$, (8.1) admits a unique solution $F \in \mathcal{A}_{\rho-\delta}$ with $\overline{F} = 0$, and one has the estimates

$$\|F\|_{\rho-\delta} \leq \frac{\varpi}{\Gamma \delta^{2n}} \|G\|_\rho, \quad \left\| \frac{\partial F}{\partial q} \right\|_{\rho-\delta} \leq \frac{\varpi}{\Gamma \delta^{2n+1}} \|G\|_\rho,$$

where $\varpi = 2^{4n+1} \left(\frac{n+1}{e}\right)^{n+1}$. Proof see appendix D.

9 P.D.E on a Torus with Time Dependent Coefficients

Consider the functions $F(q, t)$ and $G(q, t)$ where $q = (q_1, \dots, q_n) \in \mathbb{T}^n$, $t \in \mathbb{R}$. We will require as before complex extensions of functions originally defined on \mathbb{R}^{n+1} to \mathbb{C}^{n+1} . We define the complex extension of $\mathbb{T}^n \times \mathbb{R}$

$$D_{\rho, \sigma} = \{(q, t) \in \mathbb{C}^{n+1} \mid \operatorname{Re} q \in \mathbb{R}^n \bmod 2\pi, \|\operatorname{Im} q\| \leq \rho, |\operatorname{Im} t| \leq \sigma\},$$

where $\operatorname{Re} q \equiv (\operatorname{Re} q_1, \dots, \operatorname{Re} q_n)$, $\operatorname{Im} q \equiv (\operatorname{Im} q_1, \dots, \operatorname{Im} q_n)$. We define $\mathcal{A}_{\rho, \sigma}$, the set of all complex, continuous functions defined on $D_{\rho, \sigma}$ that are analytic in the interior of $D_{\rho, \sigma}$ and real for real values of the variables.

We want show that for a given function $G(q, t) \in \mathcal{A}_{\rho, \sigma}$ satisfying certain conditions there exists an $F(q, t)$ which satisfies the following

$$\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i}(q, t) = G(q, t), \quad (9.1)$$

where we identify q_{n+1} with t and set $\lambda_{n+1} = 1$.

Theorem 9.1

Given

$$\sum_{i=1}^{n+1} \lambda_i \frac{\partial F}{\partial q_i}(q, t) = G(q, t),$$

$$G(q, t) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q}, \quad F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q},$$

where $G \in \mathcal{A}_{\rho, \sigma}$ for $1 > \rho > 0$, $1 > \sigma > 0$ and $q \in \mathbb{T}^n$, $t = t_R + it_I \in \mathbb{C}$, $t_R, t_I \in \mathbb{R}$. Since $G \in \mathcal{A}_{\rho, \sigma}$ it follows $g_k(t)$ is analytic in the strip $|t_I| < \sigma$ and further assume for any given k

$$g_k(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R}) & (t_R \rightarrow -\infty) \end{cases}$$

where $\nu, \mu > 0$. Then there exist $\delta > 0$ and $\gamma > 0$, $\gamma < \nu$, $\gamma < \mu$ such that $f_k(t)$ is analytic in the strip $|t_I| < \sigma - \delta$ and satisfies

$$f_k(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R})\mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R})\mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow -\infty) \end{cases}$$

Proof

Since $F(q, t)$ and $G(q, t)$ are 2π periodic in each q_1, \dots, q_n , one can write the Fourier series

$$F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q}, \quad G(q, t) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q}.$$

Substituting these in (9.1) results in the following differential equation which the Fourier coefficients $f_k(t)$ and $g_k(t)$ must satisfy

$$i(\tilde{\lambda} \cdot k) f_k(t) + \frac{df_k(t)}{dt} = g_k(t).$$

The solution of the differential equation is given by

$$f_k(t) = f_k(0) e^{-i(\tilde{\lambda} \cdot k)t} + \int_0^t g_k(s) e^{i(\tilde{\lambda} \cdot k)(s-t)} ds. \quad (9.2)$$

We can argue $f_k(t)$ has a Fourier transform provided $f_k(0)$ satisfies certain condition. Ultimately, with this condition on $f_k(0)$, we will obtain a new expression for $f_k(t)$ which will be better suited to apply Fourier theory. Assuming that $g_k(t)$ has a complex Fourier transform $\mathcal{G}(\omega)$, with $\omega = u + iv$, $u, v \in \mathbb{R}$, analytic in some strip $-\mu < -\beta < v < \beta < \nu$, rewrite (9.2) in terms of the Fourier inversion formula. That is, given the Fourier transform

$$\mathcal{G}_k(\omega) = \int_{-\infty}^{\infty} g_k(t) e^{-i\omega t} dt, \quad (9.3)$$

and the Fourier inversion formula

$$g_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \mathcal{G}_k(\omega) e^{i\omega t} d\omega,$$

(9.2) becomes

$$f_k(t) = f_k(0) e^{-i(\tilde{\lambda} \cdot k)t} + \int_0^t \left(\int_{-\infty+i\beta}^{\infty+i\beta} \mathcal{G}_k(\omega) e^{i\omega s} d\omega \right) e^{i(\tilde{\lambda} \cdot k)(s-t)} ds.$$

Interchanging integrals and integrating with respect to s gives the expression

$$f_k(t) = f_k(0) e^{-i(\tilde{\lambda} \cdot k)t} + e^{-i(\tilde{\lambda} \cdot k)t} \int_{-\infty+i\beta}^{\infty+i\beta} \mathcal{G}_k(\omega) \left[\frac{e^{i(\omega + \tilde{\lambda} \cdot k)t} - 1}{i(\omega + \tilde{\lambda} \cdot k)} \right] d\omega. \quad (9.4)$$

We consider an arbitrary function $s(\omega)$ with the property $\int_{-\infty+i\beta}^{\infty+i\beta} s(\omega) d\omega = 1$ and rewrite (9.4) as follows

$$\begin{aligned} f_k(t) &= e^{-i(\tilde{\lambda} \cdot k)t} \int_{-\infty+i\beta}^{\infty+i\beta} \left(f_k(0) s(\omega) + \mathcal{G}_k(\omega) \left[\frac{e^{i(\omega + \tilde{\lambda} \cdot k)t} - 1}{i(\omega + \tilde{\lambda} \cdot k)} \right] \right) d\omega \\ &= \int_{-\infty+i\beta}^{\infty+i\beta} \left(f_k(0) s(\omega) e^{-i(\tilde{\lambda} \cdot k + \omega)t} + \mathcal{G}_k(\omega) \left[\frac{1 - e^{-i(\omega + \tilde{\lambda} \cdot k)t}}{i(\omega + \tilde{\lambda} \cdot k)} \right] \right) e^{i\omega t} d\omega. \end{aligned} \quad (9.5)$$

Note (9.5) is written in the form of the Fourier inversion formula. That is

$$f_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \mathcal{F}_k(\omega) e^{i\omega t} d\omega, \quad (9.6)$$

where \mathcal{F}_k the Fourier transform of $f_k(t)$ given by

$$\mathcal{F}_k(\omega) = \int_{-\infty}^{\infty} f_k(t) e^{-i\omega t} dt.$$

Consequently, from (9.5) and (9.6) we write

$$\mathcal{F}_k(\omega) = f_k(0)s(\omega)e^{-i(\tilde{\lambda} \cdot k + \omega)t} + \mathcal{G}_k(\omega) \left[\frac{1 - e^{-i(\omega + \tilde{\lambda} \cdot k)t}}{i(\omega + \tilde{\lambda} \cdot k)} \right]. \quad (9.7)$$

Furthermore, note \mathcal{F}_k is only a function of ω which implies the following

$$f_k(0)s(\omega)e^{-i(\tilde{\lambda} \cdot k + \omega)t} = \mathcal{G}_k(\omega) \frac{e^{-i(\omega + \tilde{\lambda} \cdot k)t}}{i(\omega + \tilde{\lambda} \cdot k)}, \quad (9.8)$$

or

$$f_k(0)s(\omega) = \frac{\mathcal{G}_k(\omega)}{i(\omega + \tilde{\lambda} \cdot k)}, \quad (9.9)$$

and (9.5) reduces to the following

$$f_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega) e^{i\omega t}}{i(\omega + \tilde{\lambda} \cdot k)} d\omega. \quad (9.10)$$

Note the condition on $f_k(0)$ given by (9.9) reduces to (9.10) for $t = 0$ by integrating (9.9) on both sides

$$\begin{aligned} \int_{-\infty+i\beta}^{\infty+i\beta} f_k(0)s(\omega) d\omega &= \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\omega + \tilde{\lambda} \cdot k)} d\omega, \\ f_k(0) \int_{-\infty+i\beta}^{\infty+i\beta} s(\omega) d\omega &= \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\omega + \tilde{\lambda} \cdot k)} d\omega, \\ f_k(0) &= \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\omega + \tilde{\lambda} \cdot k)} d\omega. \end{aligned}$$

Now, using Lemma E.3 it follows

$$g_k(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R}) \mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R}) \mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow -\infty) \end{cases}$$

or

$$|g_k(t)| \leq C_1 e^{-(\nu-\varepsilon)t_R} e^{-|k|\rho}, \quad 0 < c_1 < t_R < \infty,$$

$$|g_k(t)| \leq C_2 e^{(\mu-\varepsilon)t_R} e^{-|k|\rho}, \quad -\infty < t_R < c_2 < 0.$$

where the k order will pass through the proofs of Theorem E.1 and Theorem E.2 like a constant. By Theorem E.1, there exists a $\delta > 0$ such that the complex Fourier transform of $g_k(t)$ defined by

$$\mathcal{G}_k(\omega) = \int_{-\infty+i\beta}^{\infty+i\beta} g_k(t) e^{-i\omega t} dt,$$

is analytic in the strip $-\mu + \delta \leq v \leq \nu - \delta$ and satisfies

$$|\mathcal{G}_k(\omega)| \leq \frac{2}{\sqrt{2\pi}} e^{-(\sigma-\varepsilon)u} e^{-|k|\rho} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right], \quad 0 \leq u < \infty,$$

$$|\mathcal{G}_k(\omega)| \leq \frac{2}{\sqrt{2\pi}} e^{(\sigma-\varepsilon)u} e^{-|k|\rho} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right], \quad -\infty < u \leq 0$$

where we have used the fact

$$\int_0^{c_1} |g_k(\zeta - i\eta)| e^{(\nu-\delta)\zeta} d\zeta \leq C_1 c_1,$$

$$\int_{c_2}^0 |g_k(\zeta - i\eta)| e^{-(\mu-\delta)\zeta} d\zeta \leq C_2 |c_2|,$$

and $c_3 = \min(c_1, |c_2|)$, $C_3 = \max(C_1, C_2)$. From (9.7) and (9.8) we see that the complex Fourier transform of $f_k(t)$ is given by

$$\mathcal{F}_k(\omega) = \frac{\mathcal{G}_k(\omega)}{i(\tilde{\lambda} \cdot k + \omega)}.$$

We will show that $\mathcal{F}_k(\omega)$ is exponentially small with respect to u and k and analytic in two parallel strips. Clearly, since $\tilde{\lambda} \cdot k \in \mathbb{R}$, we can pick a $\gamma > 0$ such that $|\tilde{\lambda} \cdot k + \omega| > \gamma$ and since $\mathcal{G}_k(\omega)$ is analytic in the strip $-\mu < v < \nu$, $\mathcal{F}_k(\omega)$ will be analytic in the strips $A(\gamma < v < \nu)$ and $B(-\mu < v < -\gamma)$. We have

$$f_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\tilde{\lambda} \cdot k + \omega)} e^{i\omega t} d\omega,$$

and the integral converges uniformly for $|t_I| < \sigma$. Hence $f_k(t)$ is analytic in this strip. Furthermore

$$f_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\tilde{\lambda} \cdot k + \omega)} e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \frac{\mathcal{G}_k(u + i\beta)}{i(\tilde{\lambda} \cdot k + u + i\beta)} e^{i(u+i\beta)t} du,$$

and

$$\begin{aligned} |f_k(t)| &\leq \int_{-\infty}^{\infty} \frac{|\mathcal{G}_k(u + i\beta)|}{|i(\tilde{\lambda} \cdot k + u + i\beta)|} \left| e^{i(u+i\beta)(t_R + it_I)} \right| du \\ &\leq \frac{1}{\gamma} \int_{-\infty}^{\infty} |\mathcal{G}_k(u + i\beta)| e^{iut_R} |e^{-ut_I}| e^{-\beta t_R} |e^{-i\beta t_I}| du \\ &\leq \frac{2}{\sqrt{2\pi}\gamma} e^{-|k|\rho} e^{-\beta t_R} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left[\int_0^{\infty} e^{-(\sigma-\varepsilon)u} e^{-ut_I} du + \int_{-\infty}^0 e^{(\sigma-\varepsilon)u} e^{-ut_I} du \right] \\ &= \frac{2}{\sqrt{2\pi}\gamma} e^{-|k|\rho} e^{-\beta t_R} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left[\frac{e^{-(\sigma-\varepsilon+t_I)u}}{-(\sigma-\varepsilon+t_I)} \Big|_0^{\infty} + \frac{e^{(\sigma-\varepsilon-t_I)u}}{(\sigma-\varepsilon-t_I)} \Big|_{-\infty}^0 \right], \end{aligned}$$

where we have used the fact that we can pick a $\gamma > 0$ such that $|i(\tilde{\lambda} \cdot k + u + i\beta)| = [(\tilde{\lambda} \cdot k + u)^2 + \beta^2]^{1/2} \geq \beta \geq \gamma$. For the u intervals $[0, \infty)$ and $(-\infty, 0]$, we pick $t_I = -\sigma + \delta$ and $t_I = \sigma - \delta$, $\delta > \varepsilon$, respectively, so that

$$|f_k(t)| \leq \frac{4}{\sqrt{2\pi}\gamma} \frac{e^{-|k|\rho} e^{-\beta t_R}}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right].$$

Finally, we obtain the order results by taking β , which was arbitrary from the definition of $\mathcal{G}(\omega)$, arbitrarily close to ν and $-\mu$. Note that since $f_k(t)$ is only defined in the strips $A(\gamma < v < \nu)$ and $B(-\mu < v < -\gamma)$, β must be within these strips; $\gamma < \beta < \nu$ and $-\mu < -\beta < -\gamma$. By Lemma A.2 for some $\tilde{\delta} < \rho$

$$\|F\|_{\rho-\tilde{\delta}, \sigma-\delta} \leq \frac{4}{\gamma\sqrt{2\pi}} \frac{1}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left(\frac{4}{\tilde{\delta}} \right)^n.$$

We can easily find estimates for the partial derivatives of $F(q, t)$. Given

$$F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q}, \quad F \in \mathcal{A}_{\rho, \sigma},$$

the partial derivatives of $F(q, t)$ with respect to q_j $j \neq n+1$ are as follow

$$\frac{\partial F}{\partial q_j} = \sum_{k \in \mathbb{Z}^n} ik_j f_k(t) e^{ik \cdot q}.$$

For some positive $\tilde{\delta} < \rho$

$$\begin{aligned} |ik_j f_k(t)| &= |k_j| |f_k(t)| \leq \left(\frac{1}{e^{\tilde{\delta}}} \right) e^{|k|\tilde{\delta}} |f_k(t)| \\ &\leq \left(\frac{1}{e^{\tilde{\delta}}} \right) e^{|k|\tilde{\delta}} \frac{4}{\gamma\sqrt{2\pi}} \frac{e^{-|k|\rho}}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right]. \end{aligned}$$

By Lemma A.2 with $\rho - \tilde{\delta}$ in place of ρ and $\tilde{\delta}$ in place of δ we obtain the following

$$\left\| \frac{\partial F}{\partial q_i} \right\|_{\rho-\tilde{\delta}-\tilde{\delta}, \sigma-\delta} \leq \frac{4}{\gamma\sqrt{2\pi}} \frac{1}{e^{\tilde{\delta}}} \frac{1}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left(\frac{4}{\tilde{\delta}} \right)^n,$$

and $\frac{\partial F}{\partial q_i} \in \mathcal{A}_{\rho-\tilde{\delta}-\tilde{\delta}, \sigma}$. Similarly, we obtain an estimate for the partial derivative of $F(q, t)$ with respect to $q_{n+1} = t$ which is given by the following expression

$$\frac{\partial F}{\partial t} = \sum_{k \in \mathbb{Z}^n} \frac{df_k(t)}{dt} e^{ik \cdot q}.$$

Using Cauchy's Inequalities

$$\left| \frac{df_k(t)}{dt} \right| \leq \frac{1}{\delta} \|f_k\|_{\sigma} \leq \frac{1}{\delta} \frac{4}{\gamma\sqrt{2\pi}} \frac{e^{-|k|\rho}}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right].$$

By Lemma A.2 for some $\tilde{\delta} < \rho$

$$\left\| \frac{\partial F}{\partial t} \right\|_{\rho-\tilde{\delta}, \sigma-2\delta} \leq \frac{1}{\delta} \frac{4}{\gamma\sqrt{2\pi}} \frac{1}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left(\frac{4}{\tilde{\delta}} \right)^n,$$

and

$$\frac{\partial F}{\partial q_i} \in \mathcal{A}_{\rho-\tilde{\delta}, \sigma-\delta} \quad \text{for } i = 1, \dots, n+1.$$

Let $d = \max(1/e^{\tilde{\delta}}, 1/\delta)$, then

$$\left\| \frac{\partial F}{\partial q'} \right\|_{\rho-\tilde{\delta}-\tilde{\delta}, \sigma-2\delta} \leq \frac{4d}{\gamma\sqrt{2\pi}} \frac{1}{(\delta-\varepsilon)} \left[Cc_3 + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right] \left(\frac{4}{\tilde{\delta}} \right)^n.$$

□

10 Iterative Lemma

Lemma 10.1 (Iterative Lemma)

Given positive constants $\Gamma, \rho, \sigma, \kappa < 1$, consider the Hamiltonian $H(p', q') = U(p', q') + P(p', q')$ defined by

$$\begin{aligned} U(p', q') &= a + \lambda \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j}(q') p'_i p'_j + R(p', q'), \\ P(p', q') &= \kappa \left(A(q') + \sum_i B_i(q') p'_i \right), \end{aligned}$$

with $\|H\|_{\rho,\sigma} < 1$, $H \in \mathcal{A}_{\rho,\sigma}$, with some constant vector $\lambda = (\tilde{\lambda}, 1) \in \mathbb{C}^{n+1}$ such that $|\tilde{\lambda}| < L_0$ for some constant L_0 , $\tilde{\lambda} \in \Omega_\Gamma$, $A, B_i, C_{ij}, R \in \mathcal{A}_{\rho,\sigma}$ and R is of order $\|p\|^3$. Assume H is of $(C_1, C_2, c_1, c_2, \nu, \mu)p', q'$ -exponential form. Assume there are positive constants $m, f, E_1 < 1$ such that

$$2m\|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n,$$

$$f\|v\| \leq \left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\|Cv\|_{\rho,\sigma} < m^{-1}\|v\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\max(\|A\|_{\rho,\sigma}, \|B\|_{\rho,\sigma}) < E_1.$$

One can construct positive constants $\rho_*, \sigma_*, m_*, L, c_3, \gamma$ with $\rho_* < \rho, \sigma_* < \sigma, m_* < m, L = \frac{3}{2}L_0, c_3 = \min(c_1, |c_2|)$ and

$$\eta = \frac{\Lambda}{f^2 m^2 \delta^\aleph} \kappa E_1, \quad \aleph = 10n + 9, \quad \Lambda = \frac{3^n 2^{2n+12} \varpi^3 (6n+6)^4 (c_3+1)^2}{\rho_* \Gamma^3 \gamma^2} e^{\nu+\mu}, \quad \varpi = 2^{4n+1} \left(\frac{n+1}{e} \right)^{n+1},$$

such that for any $\delta > 0$ small enough so $\rho - 4\delta > \rho_*$, $\sigma - 4\delta > \sigma_*$ and with κE_1 small enough that

$$m - \frac{4(n+1)\eta}{\sigma_*^2} > m_*, \quad f - \frac{(n+3)\eta}{\sigma_*^2} > f_*, \quad (10.1)$$

there exists a $\zeta_A \in \mathbb{R}$ and an analytic canonical change of variables, $\phi : D_{\rho-4\delta, \sigma-4\delta} \rightarrow D_{\rho,\sigma}$, $\phi \in \mathcal{A}_{\rho-4\delta, \sigma-4\delta}$, which transforms the Hamiltonian H to $H'(\mathcal{P}', \mathcal{Q}') \equiv \mathcal{U}H = (H \circ \phi)(\mathcal{P}', \mathcal{Q}')$. $H'(\mathcal{P}', \mathcal{Q}')$ is of $\mathcal{P}', \mathcal{Q}'$ exponential form and can be decomposed as the original H with primed quantities $A', B', C',$ and R'

$$H' = U' + P',$$

$$U' = a' + \lambda' \cdot \mathcal{P}' + \frac{1}{2} \sum_{i,j} C'_{ij} \mathcal{P}'_i \mathcal{P}'_j + R'(\mathcal{P}', \mathcal{Q}'),$$

$$P' = \kappa' (A'(\mathcal{Q}') + B'(\mathcal{Q}') \cdot \mathcal{P}'),$$

where

$$\begin{aligned} \lambda' &= (1 + \kappa E_1 \zeta_A) \lambda \in \Omega_\Gamma, \quad a' = \overline{\overline{H'}}(0), \quad A'(\mathcal{Q}') = H'(0, \mathcal{Q}') - a', \\ B'_i(\mathcal{Q}') &= \frac{\partial H'}{\partial \mathcal{P}'_i}(0, \mathcal{Q}') - \lambda'_i, \quad C'_{ij}(\mathcal{Q}') = \frac{\partial^2 H'}{\partial \mathcal{P}'_i \partial \mathcal{P}'_j}(0, \mathcal{Q}'), \quad R' = \mathcal{O}(\|\mathcal{P}'\|^3), \end{aligned}$$

and satisfies similar conditions with positive numbers $\rho', \sigma', m', f', \kappa', E'_1$ all less than one and L' also positive with $\|H'\|_{\rho'} \leq 1$, where $A', B'_i, C'_{ij}, R' \in \mathcal{A}_{\rho', \sigma'}$,

$$2m'\|\tilde{v}\| \leq \|\tilde{C}'^* \tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n,$$

$$f'\|v\| \leq \left\| \begin{pmatrix} \overline{C}' & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\|C'v\|_{\rho', \sigma'} < m'^{-1}\|v\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\max(\|A'\|_{\rho', \sigma'}, \|B'\|_{\rho', \sigma'}) < E'_1,$$

where

$$\rho' = \rho - 4\delta > \rho_*, \quad \sigma' = \sigma - 4\delta > \sigma_*, \quad m' = m - \frac{4(n+1)\eta}{\sigma_*^2} > m_*, \quad f' = f - \frac{(n+3)\eta}{\sigma_*^2}, \quad L' = \frac{3}{2}L_0,$$

$$\kappa' = \frac{\Lambda}{f'^2 m'^2 \delta'^8} \kappa^2, \quad E'_1 = \frac{\Lambda}{f'^2 m'^2 \delta'^8} \frac{E_1}{\sigma_*}.$$

Furthermore, one has for any function $F \in \mathcal{A}_{\rho, \sigma}$, $\|\mathcal{U}F - F\|_{\rho', \sigma'} \leq \eta\|F\|_{\rho, \sigma}$.

Proof

We will construct a canonical transformation of the Hamiltonian, with a generating function for this transformation denoted by χ . The generating function is constructed in such a way it eliminates the part of the transformed Hamiltonian that prevents the preservation of the invariant structure $p' = 0$. Recall the Hamiltonian

$$H = U + P,$$

$$U(p', q') = a + \lambda \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j}(q') p'_i p'_j + R(p', q'),$$

$$P(p', q') = \kappa \left(A(q') + \sum_i B_i(q') p'_i \right).$$

We write the transformed Hamiltonian as follows

$$H' = \mathcal{U}H = U + P + \{\chi, U\} + [\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}].$$

Alternatively, assuming analyticity in t and taking the Taylor expansion, one has the Lie series

$$\mathcal{U}H = \sum_{m=0}^{\infty} \frac{t^m}{m!} L_{\chi}^m H,$$

where we denote $L_{\chi}^0 H = H$ and $L_{\chi}^m H = \{L_{\chi}^{m-1} H, \chi\}$ for $m \geq 1$. For the m th remainder of the Lie series, we use the notation

$$r_m(H, \chi, t) = \mathcal{U}H - \sum_{l=0}^{m-1} \frac{t^l}{l!} L_{\chi}^l H = \sum_{l=m}^{\infty} \frac{t^l}{l!} L_{\chi}^l H.$$

With this notation, the transformed Hamiltonian can be expressed as follows

$$H' = \mathcal{U}H = U + P + \{\chi, U\} + r_2(U, \chi, 1) + r_1(P, \chi, 1),$$

so that $\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\} = r_2(U, \chi, 1) + r_1(P, \chi, 1)$. We choose χ so that $\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}$ is second order in the size of the perturbation and

$$P + \{\chi, U\} = c\lambda \cdot p' + O(\|p'\|^2), \quad (10.2)$$

which implies $P + \{\chi, U\}$ does not contribute to P' but it shifts the frequency by a multiple of the original frequency. Furthermore, one can show all conditions are met by a generating function, similar to the one introduced by Kolmogorov, of the form

$$\chi = X(q') + \xi \cdot q' + Y(q') \cdot p', \quad (10.3)$$

where the functions $X(q')$ and $Y(q')$ are of q' -exponential form and $\xi \in \mathbb{R}^{n+1}$. A simple calculation with the appropriate definitions of U and P gives

$$\begin{aligned} \{\chi, U\} &= \left(-\lambda \cdot \xi - \sum_i \lambda_i \frac{\partial X}{\partial q'_i} \right) + \sum_j \left[-\sum_i C_{ij}(q') \xi_i - \sum_i C_{ij}(q') \left(\frac{\partial X}{\partial q'_i} \right) \right] p'_j \\ &\quad - \sum_{i,j} \lambda_j \frac{\partial Y_i}{\partial q'_j}(q') p'_i + \mathcal{O}(\|p'\|^2). \end{aligned}$$

Equivalently

$$\begin{aligned} \{\chi, U\} &= -\left(\lambda \cdot \xi + \lambda \cdot \partial_{q'} X(q') \right) + \left(-\xi \cdot C(q') - \partial_{q'} X(q') \cdot C(q') - \lambda \cdot \partial_{q'} Y(q') \right) \cdot p' \\ &\quad + \mathcal{O}(\|p'\|^2). \end{aligned} \quad (10.4)$$

To obtain the form of (10.2) we first impose the following condition

$$-\lambda \cdot \xi - \lambda \cdot \partial_{q'} X(q') + \kappa A(q') = 0. \quad (10.5)$$

We set

$$\lambda \cdot \xi = \kappa \bar{A}, \quad (10.6)$$

so that (10.5) becomes

$$\lambda \cdot \partial_{q'} X(q') = \kappa \left(A(q') - \bar{A} \right). \quad (10.7)$$

By Lemma F.2 and the definition of $A(q')$ we know the right hand side of (10.7) consists of a quasiperiodic part and a part of exponential order with respect to time. Consequently (10.7) can be split in two problems corresponding to the quasiperiodic part and to the part of exponential order with respect to time respectively and solved as explained in section 7. By lemma 8.2 and theorem 9.1 there exists an analytic function $X(q')$ satisfying (10.7). Note, by assuming the perturbation of the Hamiltonian, mainly $H^1(p', q')$, consists of a quasiperiodic part and an exponential-order-with-respect-to-time part, we are assured X and $\partial_{q'} X$ are of the same form. Next, for some $(\zeta_A, \xi) \in \mathbb{R} \times \mathbb{R}^{n+1}$ to be determined, set $\overline{(\kappa B - C \cdot \xi - C \cdot \partial_{q'} X)} = \kappa E_1 \zeta_A \lambda$. Consequently we have the following conditions $\bar{C} \cdot \xi + \kappa E_1 \zeta_A \lambda = \kappa \bar{B} - \bar{C} \cdot \overline{\partial_{q'} X}$, $\lambda \cdot \xi = \kappa \bar{A}$, or equivalently

$$\begin{pmatrix} \bar{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} = \begin{pmatrix} \kappa \bar{B} - \bar{C} \cdot \overline{\partial_{q'} X} \\ \kappa \bar{A} \end{pmatrix}. \quad (10.8)$$

It follows from lemmas 5.3 and 5.4 there exists a solution, $(\xi, \zeta_A) \in \mathbb{R}^{n+1} \times \mathbb{R}$, for the matrix equation above.

Up to this point we have determined conditions on $X(q')$ and $(\xi, \zeta_A) \in \mathbb{R}^{n+1} \times \mathbb{R}$ to obtain the Hamiltonian

$$\begin{aligned}
H'(p', q') &= U + \kappa \zeta_A \lambda \cdot p' + \mathcal{O}(\|p'\|^2) + \left[(\kappa B(q') - C(q') \cdot \xi - C(q') \cdot \partial_{q'} X(q')) \right. \\
&\quad \left. - (\kappa \overline{B} - \overline{C} \cdot \xi - \overline{C} \cdot \overline{\partial_{q'} X}) \right] \cdot p' + \left(-\lambda \cdot \partial_{q'} Y(q') \right) \cdot p' \\
&\quad + [\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}] \\
&= \tilde{U} + \beta \cdot p' + \left(-\lambda \cdot \partial_{q'} Y(q') \right) \cdot p' + \mathcal{R}_A,
\end{aligned}$$

where we have set

$$\begin{aligned}
\tilde{U} &= U + \kappa E_1 \zeta_A \lambda \cdot p' + \mathcal{O}(\|p'\|^2) \\
&= a + (1 + \kappa E_1 \zeta_A) \lambda \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j}(q') p'_i p'_j + R(p', q') + \mathcal{O}(\|p'\|^2) \\
&= a + \lambda' \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j}(q') p'_i p'_j + R(p', q') + \mathcal{O}(\|p'\|^2),
\end{aligned}$$

where $R(p', q')$ is $\mathcal{O}(\|p'\|^3)$, $\beta = \left[(\kappa B(q') - C(q') \cdot \xi - C(q') \cdot \partial_{q'} X(q')) - (\kappa \overline{B} - \overline{C} \cdot \xi - \overline{C} \cdot \overline{\partial_{q'} X}) \right]$, $\lambda' = (1 + \kappa E_1 \zeta_A) \lambda$ and

$$\mathcal{R}_A = \{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}. \quad (10.9)$$

Next we set $\left(-\lambda \cdot \partial_{q'} Y(q') + \beta \right) \cdot p' = 0$ or equivalently

$$\lambda \cdot \partial_{q'} Y(q') = \beta. \quad (10.10)$$

Note, by the definition of B and C and by the form of $\partial_{q'} X$, β consists of a quasiperiodic part and a part of exponential order with respect to time. Consequently, (10.10) can be split into two problems and solved as explained in section 7. The final Hamiltonian after one application of the generating function is therefore

$$H' = \mathcal{U}H = \left[a + \lambda' \cdot p' + \frac{1}{2} \sum_{i,j} C_{i,j}(q') p'_i p'_j + \mathcal{O}(\|p'\|^2) \right] + \mathcal{R}_A = a + \lambda' \cdot p' + \mathcal{O}(\|p'\|^2) + \mathcal{R}_A. \quad (10.11)$$

where \mathcal{R}_A is $\mathcal{O}(\kappa^2)$. By lemma F.8 the new Hamiltonian $H'(p', q')$ is of p', q' exponential form. The formula for $X(q')$ can be obtained in terms of Fourier coefficients. Consider the following

$$H^1(p', q, t) = G(p', q) + T(p', q') \in \mathcal{A}_{\rho, \sigma}, \quad (10.12)$$

$$\begin{aligned}
G(p', q) &= \sum_{k \in \mathbb{Z}^n} s_k^1(p') e^{ik \cdot q} \in \mathcal{A}_\rho, & T(p', q') &= \sum_{k \in \mathbb{Z}^n} h_k^1(p') e_k^1(t) e^{ik \cdot q} \in \mathcal{A}_{\rho, \sigma}, \\
X(q, t) &= \mathcal{Y}(q) + \mathcal{T}(q'), & \mathcal{Y}(q) &= \sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q}, & \mathcal{T}(q') &= \sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q}.
\end{aligned}$$

We have assumed $e_k^1(t)$ is analytic in the strip $-\sigma < \text{Im}(t) < \sigma$ and satisfies

$$e_k^1(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R}) & (t_R \rightarrow -\infty) \end{cases}$$

or

$$|e_k^1(t)| \leq C_0 e^{-(\nu-\varepsilon)t_R}, \quad 0 < c_1 < t_R < \infty,$$

$$|e_k^1(t)| \leq C'_0 e^{(\mu-\varepsilon)t_R}, \quad -\infty < t_R < c_2 < 0,$$

for some positive C_0, C'_0 and

$$|h_k^1(0)e_k^1(t)| \leq C_1 e^{-(\nu-\varepsilon)t_R} e^{-|k|\rho}, \quad 0 < c_1 < t_R < \infty,$$

$$|h_k^1(0)e_k^1(t)| \leq C_2 e^{(\mu-\varepsilon)t_R} e^{-|k|\rho}, \quad -\infty < t_R < c_2 < 0.$$

Recall the equation to be solved

$$\lambda \cdot \partial_{q'} X(q') = \kappa \left(A(q') - \overline{A} \right), \quad (10.13)$$

and by the defined normal form

$$\kappa A(q') = \kappa (H^1(0, q') - \overline{H^1}(0)) = \kappa \left(\sum_{k \in \mathbb{Z}^n \setminus 0} s_k^1(0) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} h_k^1(0) e_k^1(t) e^{ik \cdot q} \right).$$

Note $\overline{A} = 0$. Solving (10.13) can be done by separating the quasiperiodic part and the exponential-time-dependent part and solving the two parts separately as was presented in section four. The quasiperiodic part is as follows

$$\tilde{\lambda} \cdot \partial_{q'} \mathcal{Y}(q) = \kappa \left(G(0, q) - \overline{G}(0) \right) = \sum_{k \in \mathbb{Z}^n \setminus 0} \kappa s_k^1(0) e^{ik \cdot q}. \quad (10.14)$$

Since we assume $\tilde{\lambda} \in \Omega_\Gamma$ and since the right hand side of (10.14) has zero average, by lemma 8.2, for some positive $\delta < \rho$, (10.14) admits a unique solution

$$\mathcal{Y}(q) = \sum_{k \in \mathbb{Z}^n \setminus 0} y_k e^{ik \cdot q} \in \mathcal{A}_{\rho-\delta},$$

with $\overline{\mathcal{Y}} = 0$ and one has the estimates

$$|y_k| \leq \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \|(G - \overline{G})(0)\|_\rho e^{-|k|(\rho-\delta)}, \quad (10.15)$$

$$\|\mathcal{Y}\|_{\rho-\delta} \leq \frac{\varpi \kappa}{\Gamma \delta^{2n}} \|(G - \overline{G})(0)\|_\rho, \quad \left\| \frac{\partial \mathcal{Y}}{\partial q} \right\|_{\rho-\delta} \leq \frac{\varpi \kappa}{\Gamma \delta^{2n+1}} \|(G - \overline{G})(0)\|_\rho,$$

where $\varpi = 2^{4n+1} \left(\frac{n+1}{e} \right)^{n+1}$. The exponential-order-with-respect-to-time part of (10.13) is as follows

$$\lambda \cdot \partial_{q'} T(q') = \kappa T(0, q'), \quad (10.16)$$

which reduces to the differential equation

$$i(\tilde{\lambda} \cdot k) x_k(t) + \frac{dx_k(t)}{dt} = \kappa h_k^1(0) e_k^1(t).$$

By Theorem E.1 and lemma E.4, there exists a $\delta > 0$ such that the complex Fourier transform of $e_k^1(t)$ defined by

$$\mathcal{E}_k^1(\omega) = \int_{-\infty+i\beta}^{\infty+i\beta} e_k^1(t) e^{-i\omega t} dt,$$

exists and is analytic in the strip $-\mu + \delta \leq v \leq \nu - \delta$ and the Fourier inversion formula is

$$e_k^1(t) = \int_{-\infty}^{\infty} \mathcal{E}_k^1(\omega) e^{i\omega t} d\omega.$$

We found, (9.10), $x_k(t)$ has the following form

$$x_k(t) = \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\kappa h_k^1(0) \mathcal{E}_k^1(\omega) e^{i\omega t}}{i(\omega + \tilde{\lambda} \cdot k)} d\omega.$$

By Theorem 9.1 there exist $\delta > 0$, $\tilde{\delta} > 0$, $\tilde{\tilde{\delta}} > 0$, without loss of generality we set $\delta = \tilde{\delta} = \tilde{\tilde{\delta}}$ and $\varepsilon = \delta/2$, and $\gamma > 0$, $\gamma < \nu$, $\gamma < \mu$ such that (10.16) has a unique solution given by

$$\mathcal{T}(q, t) = \sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-\delta, \rho-\delta},$$

and the following estimates hold

$$|x_k(t)| \leq \frac{4\kappa}{\gamma\sqrt{2\pi}} \left[\frac{2C_3c_3}{\delta} + \frac{4C_3e^{-\frac{\delta}{2}c_3}}{\delta^2} \right] e^{-(\nu-\frac{\delta}{2})t_R} e^{-|k|\rho}, \quad 0 \leq t_R < \infty, \quad (10.17)$$

$$|x_k(t)| \leq \frac{4\kappa}{\gamma\sqrt{2\pi}} \left[\frac{2C_3c_3}{\delta} + \frac{4C_3e^{-\frac{\delta}{2}c_3}}{\delta^2} \right] e^{(\mu-\frac{\delta}{2})t_R} e^{-|k|\rho}, \quad -\infty < t_R < 0, \quad (10.18)$$

$$\begin{aligned} \|\mathcal{T}\|_{\rho-\delta, \sigma-\delta} &\leq \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3c_3 + \frac{2}{\delta}C_3e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n, \\ \left\| \frac{\partial \mathcal{T}}{\partial q'} \right\|_{\rho-\delta, \sigma-\delta} &\leq \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[C_3c_3 + \frac{2}{\delta}C_3e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\tilde{\delta}} \right)^n. \end{aligned}$$

Finally we have

$$\|X\|_{\rho-\delta, \rho-\delta} = \frac{\varpi\kappa}{\Gamma\delta^{2n}} \|(G - \overline{G})(0)\|_{\rho} + \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3c_3 + \frac{2}{\delta}C_3e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n,$$

$$\left\| \frac{\partial X}{\partial q'} \right\|_{\rho-\delta, \rho-\delta} = \frac{\varpi\kappa}{\Gamma\delta^{2n+1}} \|(G - \overline{G})(0)\|_{\rho} + \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[C_3c_3 + \frac{2}{\delta}C_3e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\tilde{\delta}} \right)^n.$$

Next we look at the solution of $\lambda \cdot \partial_{q'} Y = \beta$. Recall

$$\beta_j = \kappa \left(B_j(q') - \overline{B}_j \right) - \left[C(q') \cdot \partial_{q'} X(q') - \overline{C} \cdot \overline{\partial_{q'} X(q')} \right]_j - \left[C(q') \cdot \xi - \overline{C} \cdot \xi \right]_j.$$

First we have

$$B_j = \frac{\partial H^1}{\partial p'_j}(0, q') = \left[\sum_{k \in \mathbb{Z}^n} \frac{\partial s_k^1}{\partial p'_j}(0) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k^1}{\partial p'_j}(0) e_k^1(t) e^{ik \cdot q} \right] \in \mathcal{A}_{\rho-\delta, \sigma},$$

and

$$\overline{\overline{B}}_j = \frac{\partial s_0^1}{\partial p_j'}(0).$$

Substituting for the definition of B_j we obtain

$$\kappa \left(B_j(q') - \overline{\overline{B}}_j \right) = \kappa \left[\sum_{k \in \mathbb{Z}^n \setminus 0} \frac{\partial s_k^1}{\partial p_j'}(0) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k^1}{\partial p_j'}(0) e_k^1(t) e^{ik \cdot q} \right] \in \mathcal{A}_{\rho-\delta, \sigma}. \quad (10.19)$$

Next, we write out an expression for $C(q') \cdot \partial_{q'} X(q') - \overline{\overline{C}} \cdot \overline{\overline{\partial_{q'} X(q')}}$ given the following expressions

$$X(q, t) = \mathcal{Y}(q) + \mathcal{T}(q') \in \mathcal{A}_{\rho-\delta, \sigma-\delta},$$

$$\mathcal{Y}(q) = \sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q} \in \mathcal{A}_{\rho-\delta}, \quad \mathcal{T}(q') = \sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-\delta, \sigma-\delta},$$

$$C_{ij}(q') = \frac{\partial^2 \tilde{H}^0}{\partial p_i' \partial p_j'}(0) + \kappa \frac{\partial^2 H^1}{\partial p_i' \partial p_j'}(0, q') \in \mathcal{A}_{\rho-\delta, \sigma},$$

$$H^1(p', q, t) = G(p', q) + T(p', q') \in \mathcal{A}_{\rho, \sigma},$$

$$G(p', q) = \sum_{k \in \mathbb{Z}^n} s_k^1(p') e^{ik \cdot q} \in \mathcal{A}_{\rho}, \quad T(p', q') = \sum_{k \in \mathbb{Z}^n} h_k^1(p') e_k^1(t) e^{ik \cdot q} \in \mathcal{A}_{\rho, \sigma}.$$

We have completely solved for $X(q, t)$. Recall from Theorem 9.1, $x_k(t)$ is of exponential order with respect to t_R . We differentiate and obtain

$$\partial_{q'} X = \left(\partial_{q_1'} X, \dots, \partial_{q_{n+1}'} X \right) \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta},$$

$$C(q') \cdot \partial_{q'} X = \left(\sum_{l=1}^{n+1} C_{1,l}(q') \partial_{q_l'} X, \dots, \sum_{l=1}^{n+1} C_{n+1,l}(q') \partial_{q_l'} X \right) \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta},$$

and

$$\begin{aligned} (C(q') \cdot \partial_{q'} X)_j &= \sum_{l=1}^{n+1} C_{j,l}(q') \partial_{q_l'} X \\ &= \sum_{l=1}^{n+1} \left[\frac{\partial^2 \tilde{H}^0}{\partial p_l' \partial p_j'}(0) + \kappa \frac{\partial^2 H^1}{\partial p_l' \partial p_j'}(0, q') \right] \partial_{q_l'} X \\ &= \sum_{l=1}^{n+1} \left[\frac{\partial^2 \tilde{H}^0}{\partial p_l' \partial p_j'}(0) + \kappa \frac{\partial^2 H^1}{\partial p_l' \partial p_j'}(0, q') \right] \partial_{q_l'} [\mathcal{Y}(q) + \mathcal{T}(q')] \\ &= \sum_{l=1}^{n+1} \left[\frac{\partial^2 \tilde{H}^0}{\partial p_l' \partial p_j'}(0) + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p_l' \partial p_j'}(0) e^{ik \cdot q} + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p_l' \partial p_j'}(0) e_k^1(t) e^{ik \cdot q} \right] \\ &\quad \cdot \partial_{q_l'} [\mathcal{Y}(q) + \mathcal{T}(q')] \\ &= \sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p_l' \partial p_j'}(0) + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p_l' \partial p_j'}(0) e^{ik \cdot q} + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p_l' \partial p_j'}(0) e_k^1(t) e^{ik \cdot q} \right] \end{aligned}$$

$$\begin{aligned}
& \cdot \partial_{q'_l} [\mathcal{Y}(q) + \mathcal{T}(q')] \\
& + \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_{n+1} \partial p'_j} (0) e^{ik \cdot q} + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_{n+1} \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \right] \\
& \cdot \partial_{q'_{n+1}} [\mathcal{Y}(q) + \mathcal{T}(q')] \\
& = \sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \partial_{q'_l} \mathcal{Y}(q) + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} \partial_{q'_l} \mathcal{Y}(q) \right. \\
& + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \partial_{q'_l} \mathcal{Y}(q) + \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \partial_{q'_l} \mathcal{T}(q') \\
& + \left. \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} \partial_{q'_l} \mathcal{T}(q') + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \partial_{q'_l} \mathcal{T}(q') \right] \\
& + \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) \partial_{q'_{n+1}} \mathcal{T}(q') + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_{n+1} \partial p'_j} (0) e^{ik \cdot q} \partial_{q'_{n+1}} \mathcal{T}(q') \right. \\
& + \left. \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_{n+1} \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \partial_{q'_{n+1}} \mathcal{T}(q') \right].
\end{aligned}$$

At this point we will pause to examine the terms obtained in the expression above. We begin with the first term

$$\sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \partial_{q'_l} \mathcal{Y}(q) \right] = \sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \sum_{k \in \mathbb{Z}^n} ik_l y_k e^{ik \cdot q} \right] = \sum_{k \in \mathbb{Z}^n} \left[\sum_{l=1}^n ik_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \right] e^{ik \cdot q}$$

The second term

$$\begin{aligned}
\sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} \partial_{q'_l} \mathcal{Y}(q) \right] &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} ik_l y_k e^{ik \cdot q} \right] \\
&= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j} (0) k_l y_k e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&= \sum_{w \in \mathbb{Z}^n} \left[\kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j} (0) (w - m)_l y_{w-m} \right) \right] e^{iw \cdot q}.
\end{aligned}$$

The third term

$$\begin{aligned}
\sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \partial_{q'_l} \mathcal{Y}(q) \right] &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} ik_l y_k e^{ik \cdot q} \right] \\
&= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) k_l y_k e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&= \sum_{w \in \mathbb{Z}^n} \left[\sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \kappa \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) (w - m)_l y_{w-m} \right) \right] e^{iw \cdot q}.
\end{aligned}$$

The fourth term

$$\sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \partial_{q'_l} \mathcal{T}(q') \right] = \sum_{l=1}^n \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \sum_{k \in \mathbb{Z}^n} i k_l x_k(t) e^{ik \cdot q} \right] = \sum_{k \in \mathbb{Z}^n} \left[\sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right] e^{ik \cdot q}.$$

The fifth term

$$\begin{aligned} \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) e^{ik \cdot q} \partial_{q'_l} \mathcal{T}(q') \right] &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_l x_k(t) e^{ik \cdot q} \right] \\ &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) k_l x_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] \\ &= \sum_{w \in \mathbb{Z}^n} \left[\sum_{m \in \mathbb{Z}^n} \left(\sum_{l=1}^n i \kappa \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (w-m)_l x_{w-m}(t) \right) \right] e^{iw \cdot q}. \end{aligned}$$

The sixth term

$$\begin{aligned} \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) e_k^1(t) e^{ik \cdot q} \partial_{q'_l} \mathcal{T}(q') \right] &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) e_k^1(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_l x_k(t) e^{ik \cdot q} \right] \\ &= \sum_{l=1}^n \left[\kappa \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) k_l x_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] \\ &= \sum_{w \in \mathbb{Z}^n} \left[\sum_{m \in \mathbb{Z}^n} \left(\sum_{l=1}^n i \kappa \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (w-m)_l x_{w-m}(t) \right) \right] e^{iw \cdot q}. \end{aligned}$$

The seventh term

$$\frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \partial_{q'_{n+1}} \mathcal{T}(q') = \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n} \left(\frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_k}{dt}(t) \right) e^{ik \cdot q}.$$

The eighth term

$$\begin{aligned} \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_{n+1} \partial p'_j}(0) e^{ik \cdot q} \partial_{q'_{n+1}} \mathcal{T}(q') &= \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_{n+1} \partial p'_j}(0) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} \\ &= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} = \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_{w-m}}{dt}(t) \right) e^{iw \cdot q}. \end{aligned}$$

The ninth term

$$\begin{aligned} \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_{n+1} \partial p'_j}(0) e_k^1(t) e^{ik \cdot q} \partial_{q'_{n+1}} \mathcal{T}(q') &= \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_{n+1} \partial p'_j}(0) e_k^1(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} \\ &= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j}(0) e_m^1(t) \frac{dx_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} = \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j}(0) e_m^1(t) \frac{dx_{w-m}}{dt}(t) \right) e^{iw \cdot q}. \end{aligned}$$

We finally write

$$(C(q') \cdot \partial_{q'} X)_j = \sum_{k \in \mathbb{Z}^n} \left[\sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) + \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (k-m)_l y_{k-m} \right) \right] e^{ik \cdot q}$$

$$\begin{aligned}
& + \sum_{k \in \mathbb{Z}^n} \left[\kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) (k-m)_l y_{k-m} \right) + \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \right. \\
& + \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j} (0) (k-m)_l x_{k-m}(t) \right) \\
& + \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \\
& + \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) \frac{dx_k}{dt}(t) + \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j} (0) \frac{dx_{k-m}}{dt}(t) \\
& \left. + \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j} (0) e_m^1(t) \frac{dx_{k-m}}{dt}(t) \right] e^{ik \cdot q}
\end{aligned}$$

and

$$\begin{aligned}
& (C(q') \cdot \partial_{q'} X)_j - (\bar{C} \cdot \overline{\partial_{q'} X})_j = \sum_{k \in \mathbb{Z}^n \setminus 0} \left[\sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \right] e^{ik \cdot q} \\
& + \kappa \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j} (0) (k-m)_l y_{k-m} - i \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) k_l y_k \right) e^{ik \cdot q} \\
& + \sum_{k \in \mathbb{Z}^n} \left[\kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) (k-m)_l y_{k-m} \right) + \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \right. \\
& + \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j} (0) (k-m)_l x_{k-m}(t) \right) \\
& + \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j} (0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \\
& + \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) \frac{dx_k}{dt}(t) + \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j} (0) \frac{dx_{k-m}}{dt}(t) \\
& \left. + \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j} (0) e_m^1(t) \frac{dx_{k-m}}{dt}(t) \right] e^{ik \cdot q}. \tag{10.20}
\end{aligned}$$

Next we write out the expression $\left(C(q') \cdot \xi - \bar{C} \cdot \xi \right)_j$.

$$\begin{aligned}
(C(q') \cdot \xi)_j &= \sum_{l=1}^{n+1} C_{j,l}(q') \xi_l = \sum_{l=1}^{n+1} \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) + \kappa \frac{\partial^2 H^1}{\partial p'_l \partial p'_j} (0, q') \right] \xi_l \\
&= \sum_{l=1}^{n+1} \left[\frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} + \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \right] \xi_l
\end{aligned}$$

and

$$\left(C(q') \cdot \xi - \bar{C} \cdot \xi \right)_j = \sum_{l=1}^{n+1} \left[\kappa \xi_l \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j} (0) e^{ik \cdot q} + \kappa \xi_l \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j} (0) e_k^1(t) e^{ik \cdot q} \right]$$

$$= \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) \right) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \left(\sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) \right) e_k^1(t) e^{ik \cdot q}. \quad (10.21)$$

We have thus separated the right hand side of

$$\lambda \cdot \partial_{q'} Y = \beta, \quad (10.22)$$

into a quasiperiodic and a exponential-order-with-respect-to-time part. We first solve the quasiperiodic part. Assume

$$Y_j(q') = \mathcal{S}_j(q) + \mathcal{F}_j(q'), \quad \mathcal{S}_j(q) = \sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q}, \quad \mathcal{F}_j(q') = \sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q}.$$

The j th component of (10.22) is given by

$$\lambda \cdot \partial_{q'} Y_j(q') = \sum_{i=1}^{n+1} \lambda_i \partial_{q'_i} Y_j(q') = \beta_j. \quad (10.23)$$

From (10.19), (10.20) and (10.21) we obtain the quasiperiodic part of β_j

$$\begin{aligned} \beta_j^Q(q) &= \kappa \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{\partial s_k^1}{\partial p'_j}(0) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n \setminus 0} \left[\sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right] e^{ik \cdot q} \\ &\quad + \kappa \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) k_l y_k e^{im \cdot q} e^{ik \cdot q} - i \frac{\partial^2 s_{-k}^1}{\partial p'_l \partial p'_j}(0) k_l y_k \right) \\ &\quad + \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) \right) e^{ik \cdot q} \\ &= \sum_{k \in \mathbb{Z}^n \setminus 0} \left[\kappa \frac{\partial s_k^1}{\partial p'_j}(0) + \sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right. \\ &\quad \left. + \kappa \sum_{l=1}^n \sum_{\substack{m \in \mathbb{Z}^n \\ m \neq -k}} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) k_l y_k e^{im \cdot q} + \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) \right] e^{ik \cdot q} \\ &= \sum_{h \in \mathbb{Z}^n \setminus 0} \left[\kappa \frac{\partial s_h^1}{\partial p'_j}(0) + \sum_{l=1}^n i h_l y_h \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right. \\ &\quad \left. + \kappa \sum_{l=1}^n \sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (h-m)_l y_{h-m} + \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_h^1}{\partial p'_l \partial p'_j}(0) \right] e^{ih \cdot q} \\ &= \sum_{k \in \mathbb{Z}^n \setminus 0} \beta_{j,k}^Q e^{ik \cdot q}, \end{aligned}$$

where we have set

$$\beta_{j,k}^Q = \kappa \frac{\partial s_h^1}{\partial p'_j}(0) + \sum_{l=1}^n i h_l y_h \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) + \kappa \sum_{l=1}^n \sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (h-m)_l y_{h-m} + \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_h^1}{\partial p'_l \partial p'_j}(0).$$

The quasiperiodic part of (10.22) is

$$\tilde{\lambda} \cdot \partial_q (\mathcal{S}_j(q)) = \beta_j^Q(q) = \sum_{k \in \mathbb{Z}^n \setminus 0} \beta_{j,k}^Q e^{ik \cdot q} \in \mathcal{A}_{\rho-2\delta}. \quad (10.24)$$

Since we assume $\tilde{\lambda} \in \Omega_\Gamma$ and since the right hand side of (10.24) has zero average, by lemma 8.2, for some positive $\delta < \rho$, (10.24) admits a unique solution

$$\mathcal{S}_j(q) = \sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q} \in \mathcal{A}_{\rho-3\delta},$$

with $\overline{\mathcal{S}_j} = 0$ and one has the estimates

$$\|\mathcal{S}_j\|_{\rho-3\delta} \leq \frac{\varpi}{\Gamma \delta^{2n}} \|\beta_j^Q\|_{\rho-2\delta}, \quad \left\| \frac{\partial \mathcal{S}_j}{\partial q} \right\|_{\rho-3\delta} \leq \frac{\varpi}{\Gamma \delta^{2n+1}} \|\beta_j^Q\|_{\rho-2\delta},$$

where $\varpi = 2^{4n+1} \left(\frac{n+1}{e}\right)^{n+1}$. Next we want to solve the exponential-order-with-respect-to-time part of (10.22). From (10.19), (10.20) and (10.21) we obtain the exponential-time dependent part of β_j

$$\begin{aligned} \beta_j^E(q') &= \sum_{k \in \mathbb{Z}^n} \left[\kappa \frac{\partial h_k^1}{\partial p'_j}(0) e_k^1(t) \right. \\ &+ \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l y_{k-m} \right) + \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \\ &+ \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (k-m)_l x_{k-m}(t) \right) \\ &+ \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \\ &+ \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_k}{dt}(t) + \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_{k-m}}{dt}(t) \\ &+ \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j}(0) e_m^1(t) \frac{dx_{k-m}}{dt}(t) \\ &\left. + \left(\sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) \right) e_k^1(t) \right] e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n} \beta_{jk}^E(t) e^{ik \cdot q}. \end{aligned}$$

Therefore, the exponential-order-with-respect-to-time dependent part of (10.22) is

$$\lambda \cdot \partial_{q'} \mathcal{F}_j(q') = \beta_j^E(q') = \sum_{k \in \mathbb{Z}^n} \beta_{jk}^E(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta}. \quad (10.25)$$

To solve (10.25) we must find an exponential bound for the time dependent Fourier coefficients $\beta_{jk}^E(t)$. This exponential bound can be found by bounding each of the nine time dependent terms that make up $\beta_{jk}^E(t)$. Note, to bound the terms involving sums over m we use the bounds found in appendix H. For the first term

$$\left| \frac{\partial h_k^1}{\partial p'_j}(0) e_k^1(t) \right| \leq \frac{\kappa}{\delta} C_1 e^{-(\nu-\varepsilon)t_R} e^{-|k|\rho}, \quad 0 < c_1 < t_R < \infty,$$

$$\left| \frac{\partial h_k^1}{\partial p'_j}(0) e_k^1(t) \right| \leq \frac{\kappa}{\delta} C_2 e^{(\mu-\varepsilon)t_R} e^{-|k|\rho}, \quad -\infty < t_R < c_2 < 0.$$

For the second term we refer to (10.15)

$$\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l y_{k-m} \right) \right|$$

$$\begin{aligned}
&\leq \sum_{l=1}^n \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|h_m^1\|_\rho}{\delta^2} |e_m^1(t)(k-m)_l y_{k-m}| \\
&\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|h_m^1\|_\rho}{\delta^2} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho e^{-|k-m|(\rho-\delta)} |e_m^1(t)| |k-m| \\
&\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|h_m^1\|_\rho}{\delta^2} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho e^{-|k-m|(\rho-\delta)} \frac{e^{|k-m|\delta}}{e\delta} |e_m^1(t)| \\
&\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{1}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho e^{-|k-m|(\rho-2\delta)} \|h_m^1\|_\rho |e_m^1(t)|
\end{aligned}$$

so that

$$\begin{aligned}
&\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t)(k-m)_l y_{k-m} \right) \right| \\
&\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \frac{1}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_1 \right] e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R}, \\
&\leq \left[\kappa \frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \left(\frac{1}{1-e^{-2\delta}}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_1 \right] e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R} \\
&0 < c_1 < t_R < \infty,
\end{aligned}$$

$$\begin{aligned}
&\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t)(k-m)_l y_{k-m} \right) \right| \\
&\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \frac{1}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_2 \right] e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} e^{-(\mu-\frac{\delta}{2})t_R} \\
&\leq \left[\kappa \frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta}\right)^n \left(\frac{1}{1-e^{-2\delta}}\right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_2 \right] e^{-|k|(\rho-2\delta)} e^{-(\mu-\frac{\delta}{2})t_R}, \\
&-\infty < t_R < c_2 < 0.
\end{aligned}$$

For the third term

$$\left| \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \leq \sum_{l=1}^n \left(\frac{1}{e\delta}\right) e^{|k|\delta} \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| |x_k(t)|,$$

so that

$$\begin{aligned}
&\left| \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \\
&\leq \left[\sum_{l=1}^n \left(\frac{1}{e\delta}\right) e^{|k|\delta} \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-(\nu-\frac{\delta}{2})t_R} e^{-|k|\rho}, \quad 0 \leq t_R < \infty, \\
&\left| \sum_{l=1}^n i k_l x_k(t) \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right|
\end{aligned}$$

$$\leq \left[\sum_{l=1}^n \left(\frac{1}{e\delta} \right) e^{|k|\delta} \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \right] e^{(\mu - \frac{\delta}{2})t_R} e^{-|k|\rho}, \quad -\infty \leq t_R < 0.$$

The fourth term

$$\begin{aligned} \left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (k-m)_l x_{k-m}(t) \right) \right| &\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|s_m^1\|_\rho}{\delta^2} |k-m| |x_{k-m}(t)| \\ &\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|G\|_\rho e^{-|m|\rho}}{\delta^2} \left(\frac{1}{e\delta} \right) e^{|k-m|\delta} |x_{k-m}(t)|, \end{aligned}$$

so that

$$\begin{aligned} &\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (k-m)_l x_{k-m}(t) \right) \right| \\ &\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|(\rho-\delta)} e^{-(\nu - \frac{\delta}{2})t_R} \\ &\leq \left[3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-(\nu - \frac{\delta}{2})t_R} \end{aligned}$$

for $0 \leq t_R < \infty$,

$$\begin{aligned} &\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) (k-m)_l x_{k-m}(t) \right) \right| \\ &\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \left(\frac{\|G\|}{e\delta^3} \right) \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|(\rho-\delta)} e^{(\mu - \frac{\delta}{2})t_R} \\ &\leq \left[3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-(\mu - \frac{\delta}{2})t_R}, \end{aligned}$$

for $-\infty \leq t_R < 0$.

The fifth term

$$\begin{aligned} &\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \right| \\ &\leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|h_m^1\|_\rho}{\delta^2} |k-m| |e_m^1(t) x_{k-m}(t)| \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{1}{\delta^2} \left(\frac{1}{e\delta} \right) e^{|k-m|\delta} \|h_m^1\|_\rho |e_m^1(t) x_{k-m}(t)|, \end{aligned}$$

so that

$$\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \right|$$

$$\begin{aligned}
&\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \left(\frac{1}{e\delta^3} \right) C_1 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|(\rho-\delta)} e^{-2(\nu-\frac{\delta}{2})t_R} \\
&\leq \left[\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n C_1 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-2(\nu-\frac{\delta}{2})t_R}
\end{aligned}$$

for $0 < c_1 < t_R < \infty$,

$$\begin{aligned}
&\left| \kappa \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 h_m^1}{\partial p'_l \partial p'_j}(0) e_m^1(t) (k-m)_l x_{k-m}(t) \right) \right| \\
&\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \left(\frac{1}{e\delta^3} \right) C_2 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|(\rho-\delta)} e^{2(\mu-\frac{\delta}{2})t_R} \\
&\leq \left[\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n C_2 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-2(\mu-\frac{\delta}{2})t_R}
\end{aligned}$$

for $-\infty < t_R < c_2 < 0$.

For the sixth, seventh and eighth term we refer to lemma *F.5* when dealing with the bound of the derivative of an exponential-time dependent function. The sixth term

$$\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_k}{dt}(t) \right| \leq \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \right| \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{-(\nu-\frac{\delta}{2})t_R} e^{-|k|\rho},$$

for $0 \leq t_R < \infty$,

$$\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_k}{dt}(t) \right| \leq \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \right| \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{(\mu-\frac{\delta}{2})t_R} e^{-|k|\rho},$$

for $-\infty \leq t_R < 0$.

The seventh term

$$\left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_{k-m}}{dt}(t) \right| \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|s_m^1\|_\rho}{\delta^2} \left| \frac{dx_{k-m}}{dt}(t) \right| \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|G\|_\rho e^{-|m|\rho}}{\delta^2} \left| \frac{dx_{k-m}}{dt}(t) \right|,$$

so that

$$\begin{aligned}
&\left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j}(0) \frac{dx_{k-m}}{dt}(t) \right| \\
&\leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \|G\|_\rho \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{-(\nu-\frac{\delta}{2})t_R} e^{-|k-m|\rho} \\
&\leq \left[3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-(\nu-\frac{\delta}{2})t_R},
\end{aligned}$$

for $0 \leq t_R < \infty$,

$$\begin{aligned}
& \left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 s_m^1}{\partial p'_{n+1} \partial p'_j} (0) \frac{dx_{k-m}(t)}{dt} \right| \\
& \leq \sum_{m \in \mathbb{Z}^n} \left[\kappa \|G\|_\rho \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{(\mu-\frac{\delta}{2})t_R} e^{-|k-m|\rho} \\
& \leq \left[3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-(\mu-\frac{\delta}{2})t_R},
\end{aligned}$$

for $-\infty \leq t_R < 0$.

The eighth term

$$\left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j} (0) e_m^1(t) \frac{dx_{k-m}(t)}{dt} \right| \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{\|h_m^1\|_\rho}{\delta^2} |e_m^1(t)| \left| \frac{dx_k}{dt}(t) \right| \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{1}{\delta^2} \|h_m^1\|_\rho |e_m^1(t)| \left| \frac{dx_k}{dt}(t) \right|,$$

so that

$$\begin{aligned}
& \left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j} (0) e_m^1(t) \frac{dx_{k-m}(t)}{dt} \right| \\
& \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{1}{\delta^2} C_1 e^{-(\nu-\frac{\delta}{2})t_R} e^{-|m|\rho} \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{-(\nu-\frac{\delta}{2})t_R} e^{-|k-m|\rho} \\
& = \sum_{m \in \mathbb{Z}^n} \left[\kappa C_1 \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|\rho} e^{-2(\nu-\frac{\delta}{2})t_R} \\
& \leq \left[3^n \kappa C_1 \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-2(\nu-\frac{\delta}{2})t_R}
\end{aligned}$$

for $0 < c_1 < t_R < \infty$,

$$\begin{aligned}
& \left| \sum_{m \in \mathbb{Z}^n} \kappa \frac{\partial^2 h_m^1}{\partial p'_{n+1} \partial p'_j} (0) e_m^1(t) \frac{dx_{k-m}(t)}{dt} \right| \\
& \leq \sum_{m \in \mathbb{Z}^n} \kappa \frac{1}{\delta^2} C_1 e^{(\mu-\frac{\delta}{2})t_R} e^{-|m|\rho} \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{(\mu-\frac{\delta}{2})t_R} e^{-|k-m|\rho} \\
& = \sum_{m \in \mathbb{Z}^n} \left[\kappa C_1 \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|m|\rho} e^{-|k-m|\rho} e^{2(\mu-\frac{\delta}{2})t_R} \\
& \leq \left[3^n \kappa C_m \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right] e^{-|k|(\rho-\delta)} e^{-2(\mu-\frac{\delta}{2})t_R}
\end{aligned}$$

for $-\infty < t_R < c_2 < 0$.

The nine th term

$$\left| \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) e_k^1(t) \right| \leq \kappa |\xi| \frac{\|h_k^1\|_\rho}{\delta^2} |e_k^1(t)|,$$

so that

$$\left| \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) e_k^1(t) \right| \leq \kappa |\xi| \frac{\|h_k^1\|_\rho}{\delta^2} C_1 e^{-(\nu - \frac{\delta}{2})t_R} e^{-|k|\rho}, \quad 0 < c_1 < t_R < \infty,$$

$$\left| \sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 h_k^1}{\partial p'_l \partial p'_j}(0) e_k^1(t) \right| \leq \kappa |\xi| \frac{\|h_k^1\|_\rho}{\delta^2} C_2 e^{(\mu - \frac{\delta}{2})t_R} e^{-|k|\rho}, \quad -\infty < t_R < c_2 < 0.$$

Note we have bounded the sums on m as indicated in Appendix 3. Putting the estimates for the nine terms we obtain the following bound for β_j^E .

$$\begin{aligned} |\beta_{jk}^E(t)| &\leq \left[\frac{\kappa}{\delta} C_1 e^{-|k|\rho} \right. \\ &+ \left(\kappa \frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1 - e^{-2\delta}} \right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_1 \right) e^{-|k|(\rho - 2\delta)} e^{-(\nu - \frac{\delta}{2})t_R} \\ &+ \left(\sum_{l=1}^n \left(\frac{1}{e\delta} \right) \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{-|k|(\rho - \delta)} \right) \\ &+ \left(3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1 - e^{-2\delta}} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) e^{-|k|(\rho - \delta)} e^{-(\nu - \frac{\delta}{2})t_R} \\ &+ \left(\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1 - e^{-2\delta}} \right)^n C_1 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) e^{-|k|(\rho - \delta)} e^{-2(\nu - \frac{\delta}{2})t_R} \\ &+ \left(\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \right| \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] e^{-|k|\rho} \right) \\ &+ \left(3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) e^{-|k|(\rho - \delta)} e^{-(\nu - \frac{\delta}{2})t_R} \\ &+ \left(3^n \kappa C_1 \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) e^{-|k|(\rho - \delta)} e^{-2(\nu - \frac{\delta}{2})t_R} \\ &+ \left. \kappa |\xi| \frac{1}{\delta^2} C_1 e^{-|k|\rho} \right] e^{-(\nu - \frac{\delta}{2})t_R}, \quad 0 < c_1 < t_R < \infty, \end{aligned}$$

where we have used the fact $e^{-2(\nu - \frac{\delta}{2})t_R} \leq e^{-(\nu - \frac{\delta}{2})t_R}$. We can further simplify by using the fact $e^{-|k|\rho} \leq e^{-|k|(\rho - \delta)} \leq e^{-|k|(\rho - 2\delta)}$ so that

$$\begin{aligned} |\beta_{jk}^E(t)| &\leq \left[\frac{\kappa}{\delta} C_1 + \left(\kappa \frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1 - e^{-2\delta}} \right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_1 \right) \right. \\ &+ \left(\sum_{l=1}^n \left(\frac{1}{e\delta} \right) \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\ &+ \left. \left(3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1 - e^{-2\delta}} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n C_1 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) \right| \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(3^n \kappa C_1 \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \kappa |\xi| \frac{1}{\delta^2} C_1 \left[e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R}, \quad 0 < c_1 < t_R < \infty, \right.
\end{aligned}$$

similarly

$$\begin{aligned}
|\beta_{jk}^E(t)| & \leq \left[\frac{\kappa}{\delta} C_2 + \left(\kappa \frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1-e^{-2\delta}} \right)^n \|G(0, q) - \overline{G}(0)\|_\rho C_2 \right) \right. \\
& + \left(\sum_{l=1}^n \left(\frac{1}{e\delta} \right) \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} (0) \right| \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n C_2 \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j} (0) \right| \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left(3^n \kappa C_2 \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8\kappa}{\delta\gamma\sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2}c_3} \right] \right) \\
& + \left. \kappa |\xi| \frac{1}{\delta^2} C_2 \left[e^{-|k|(\rho-2\delta)} e^{(\mu-\frac{\delta}{2})t_R}, \quad -\infty < t_R < c_2 < 0. \right] \right.
\end{aligned}$$

At this point we must relate the constants used to bound the time dependence, C_1 and C_2 , to a constant which bounds the entire perturbation. Recall that by assumption on the time dependence and by the quasiperiodic q dependence we obtain the following estimates

$$|h_k^1(0)e_k^1(t)| \leq C_1 e^{-|k|\rho} e^{-(\nu-\frac{\delta}{2})t_R}, \quad 0 < c_1 \geq t_R < \infty,$$

$$|h_k^1(0)e_k^1(t)| \leq C_2 e^{-|k|\rho} e^{(\mu-\frac{\delta}{2})t_R}, \quad -\infty < t_R \leq c_2 < 0,$$

and clearly

$$|h_k^1(0)e_k^1(t)| \leq C_1 e^{-(\nu-\frac{\delta}{2})t_R}, \quad 0 < c_1 \geq t_R < \infty,$$

$$|h_k^1(0)e_k^1(t)| \leq C_2 e^{(\mu-\frac{\delta}{2})t_R}, \quad -\infty < t_R \leq c_2 < 0.$$

We want this estimates also to include the interval $c_2 < t_R < c_1$. The assumption on the function $e_k^1(t)$ in this interval was simply to be a bounded function. Consequently, there exists a constant E_0 such that $C_1 \leq E_0$, $C_2 \leq E_0$ and the following estimates hold

$$|h_k^1(0)e_k^1(t)| \leq E_0 e^{-|k|\rho}, \quad 0 < t_R < \infty,$$

$$|h_k^1(0)e_k^1(t)| \leq E_0 e^{-|k|\rho}, \quad -\infty < t_R \leq 0.$$

Now recall the total perturbation $H^1(0, q') = G(q) + T(q')$ and assume $|G(q)| \leq E_G$. By the argument above we have

$$\begin{aligned} |T(q')| &\leq \sum_{k \in \mathbb{Z}^n} |h_k^1(0)| |e_k^1(t)| |e^{ik \cdot q}| \leq \sum_{k \in \mathbb{Z}^n} E_0 e^{-|k|\rho} e^{|k|(\rho - \delta)} \\ &= \sum_{k \in \mathbb{Z}^n} E_0 e^{-|k|\delta} = 2E_0 \frac{1}{1 - e^{-\delta}} = E_T. \end{aligned}$$

Finally we have

$$C_1 \leq E_0 \leq E_T \leq E_G + E_T = E \leq E_1,$$

$$C_2 \leq E_0 \leq E_T \leq E_G + E_T = E \leq E_1,$$

$$C_3 = \max(C_1, C_2) \leq E_1,$$

where E_1 is the bound on the total perturbation $H^1(0, q')$. The estimates on $\beta_{jk}^E(t)$ become

$$\begin{aligned} |\beta_{jk}^E(t)| &\leq \left[\frac{1}{\delta} + \left(\frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1 - e^{-2\delta}} \right)^n \|G(0, q) - \overline{G}(0)\|_\rho \right) \right. \\ &\quad + \left(\sum_{l=1}^n \left(\frac{1}{e\delta} \right) \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left(3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1 - e^{-2\delta}} \right)^n \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left(\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1 - e^{-2\delta}} \right)^n C_1 \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left(\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \right| \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta} \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left(3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta^3} \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left(3^n \kappa C_1 \frac{e^{(\nu - \frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\ &\quad + \left. \left| \xi \right| \frac{1}{\delta^2} \right] \kappa E_1 e^{-|k|(\rho - 2\delta)} e^{-(\nu - \frac{\delta}{2})t_R}, \quad 0 < c_1 < t_R < \infty, \end{aligned}$$

similarly

$$|\beta_{jk}^E(t)| \leq \left[\frac{1}{\delta} + \left(\frac{3^n}{e\delta^3} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1 - e^{-2\delta}} \right)^n \|G(0, q) - \overline{G}(0)\|_\rho \right) \right.$$

$$\begin{aligned}
& + \left(\sum_{l=1}^n \left(\frac{1}{e\delta} \right) \left| \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) \right| \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + \left(3^n \kappa \left(\frac{\|G\|_\rho}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + \left(\kappa \left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n C_2 \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + \left(\left| \frac{\partial^2 \tilde{H}^0}{\partial p'_{n+1} \partial p'_j}(0) \right| \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta} \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + \left(3^n \kappa \|G\|_\rho \left(\frac{1}{e\delta} \right)^n \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + \left(3^n \kappa C_2 \frac{e^{(\mu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8}{\delta \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \right) \\
& + |\xi| \frac{1}{\delta^2} \left[\kappa E_1 e^{-|k|(\rho-2\delta)} e^{(\mu-\frac{\delta}{2})t_R}, \quad -\infty < t_R < c_2 < 0. \right]
\end{aligned}$$

We take a pause to address the analyticity issue. We know from the functions that make up $\beta_j^E(q')$ that $\beta_j^E(q') \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta}$. Our estimates above demonstrate the Fourier coefficients of $\beta_j^E(q')$ depend on k as $e^{-|k|(\rho-2\delta)}$ which agrees by Lemma A.1 with our expectations. Next we want to further simplify the estimates above by removing dependence on C_1 , C_2 , G , κ , \tilde{H}^0 , and ξ . We first find an estimate for $\|\xi\|$. Recall from Lemma 5.4 there exists a positive constant f such that

$$\left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} \right\| \geq f \left\| \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} \right\| \geq f \|\xi\|.$$

Also from (10.8) we have

$$\begin{aligned}
& \left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} \right\| = \left\| \begin{pmatrix} \kappa \overline{B} - \overline{C} \cdot \overline{\partial_{q'} X} \\ \kappa \overline{A} \end{pmatrix} \right\| \\
& \leq \kappa \|B\|_{\rho, \sigma} + \|C \cdot \partial_{q'} X\|_{\rho, \sigma} + \kappa \|A\|_{\rho, \sigma} \leq 2\kappa E_1 + m^{-1} \|\partial_{q'} X\|_{\rho, \sigma} \\
& \leq 2\kappa E_1 + \frac{\varpi \kappa}{m \Gamma \delta^{2n+1}} \|G(0, q) - \overline{G}(0)\|_\rho + \frac{8\kappa}{m \delta^2 \gamma \sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta}{2} c_3} \right] \left(\frac{4}{\delta} \right)^n.
\end{aligned}$$

We finally obtain

$$\|\xi\| \leq \frac{1}{f} \left[2\kappa E_1 + \frac{2\varpi \kappa}{m \Gamma \delta^{2n+1}} E_1 + \frac{8\kappa}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} \right] \left(\frac{4}{\delta} \right)^n E_1 \right].$$

With this bound we can bound $|\xi| < (n+1)\|\xi\|$. Also note

$$\|G(0, q) - \overline{G}(0)\|_\rho \leq 2\|G(0, q)\|_\rho \leq 2\|H^1\|_\rho \leq 2E_1, \quad \|G(0, q)\|_\rho \leq E_1.$$

The bounds on ξ , G together with $\kappa C_1 \leq \kappa E_1 < 1$, $\kappa C_2 \leq \kappa E_1 < 1$, and

$$\sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \leq \frac{n}{\delta} \|H\|_\rho \leq \frac{n}{\delta},$$

are used to reduce the bounds on $|\beta_{jk}^E(t)|$ to the following

$$\begin{aligned}
|\beta_{jk}^E(t)| &\leq \left[\frac{1}{\delta} + \left(\frac{3^n}{e\delta^3} \frac{1}{\Gamma} \left(\frac{n}{e\delta} \right)^n \left(\frac{1}{1-e^{-2\delta}} \right)^n \right) + \left(\left(\frac{1}{e\delta} \right) \frac{n}{\delta} \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \right. \\
&\quad + \left(3^n \left(\frac{1}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \\
&\quad + \left(\left(\frac{3^n}{e\delta^3} \right) \left(\frac{1}{1-e^{-2\delta}} \right)^n \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \\
&\quad + \left(\frac{1}{\delta} \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta} \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \\
&\quad + \left(3^n \left(\frac{1}{e\delta} \right)^n \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \\
&\quad + \left(3^n \frac{e^{(\nu-\frac{\delta}{2})\delta}}{\delta^3} \left(\frac{1}{e\delta} \right)^n \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \right) \\
&\quad + \frac{1}{\delta^2} \frac{(n+1)}{f} \left[2 + \frac{2\varpi}{m\Gamma\delta^{2n+1}} + \frac{8}{\delta\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n \right] \\
&\quad \cdot \kappa E_1 e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R} \\
&\leq \frac{3^n 16(n+1)\varpi}{\delta^{4n+3}\Gamma\gamma f m} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \frac{e^{(\nu-\frac{\delta}{2})\delta}}{(1-e^{-\delta})^n} \kappa E_1 e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R}
\end{aligned}$$

for $0 < c_1 < t_R < \infty$.

Now we use the fact for $x \geq 0$, $e^{-x} < x + 1$. Setting $x = \frac{\delta}{2}c_3$ we have

$$c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} < c_3 + \frac{2}{\delta} \left(\frac{\delta}{2}c_3 + 1 \right) = 2c_3 + \frac{2}{\delta} < \frac{2}{\delta}(c_3 + 1).$$

So that

$$|\beta_{jk}^E(t)| \leq \frac{3^n 32(n+1)\varpi}{\delta^{4n+4}\Gamma\gamma f m} (c_3 + 1) \frac{e^{(\nu-\frac{\delta}{2})\delta}}{(1-e^{-\delta})^n} \kappa E_1 e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R},$$

for $0 < c_1 < t_R < \infty$.

We further simplify by using the fact $\delta < 1$ and the following

$$\frac{1}{1-e^{-2\delta}} < \frac{1}{1-e^{-\delta}} < \frac{1}{\delta} + 1 < \frac{2}{\delta}, \quad e^{(\nu-\varepsilon)\delta} < e^{\nu-\varepsilon}.$$

The exponential bound of $\beta_{jk}^E(t)$ is

$$|\beta_{jk}^E(t)| \leq \frac{6^n 32(n+1)\varpi}{\delta^{5n+4}\Gamma\gamma f m} (c_3 + 1) e^\nu \kappa E_1 e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R} = C^{\beta_1} \kappa E_1 e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{\delta}{2})t_R},$$

for $0 < c_1 \leq t_R < \infty$ and

$$|\beta_{jk}^E(t)| \leq \frac{6^n 32(n+1)\varpi}{\delta^{5n+4}\Gamma\gamma f m} (c_3 + 1) e^\mu \kappa E_1 e^{-|k|(\rho-2\delta)} e^{(\mu-\frac{\delta}{2})t_R} = C^{\beta_2} \kappa E_1 e^{-|k|(\rho-2\delta)} e^{(\mu-\frac{\delta}{2})t_R}.$$

for $-\infty < t_R \leq c_2 < 0$. Next, by Theorem 9.1 there exists $\delta > 0, \tilde{\delta} > 0, \tilde{\tilde{\delta}} > 0$, without loss of generality we set $\delta = \tilde{\delta} = \tilde{\tilde{\delta}}$ and $\varepsilon = \delta/2$, and $\gamma < \nu, \gamma < \mu$ such that (10.25) has a unique solution given by

$$\mathcal{F}_j(q') = \sum_{k \in \mathbb{Z}^n} \mathcal{F}_{jk}(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-3\delta, \sigma-3\delta}.$$

Note, since

$$\sum_{k \in \mathbb{Z}^n} \beta_{jk}^E(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta},$$

when using the bounds obtained in theorem 9.1 we replace δ with 3δ and the following estimates hold

$$\begin{aligned} |\mathcal{F}_{jk}(t)| &\leq \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} e^{-|k|(\rho-2\delta)} e^{-(\nu-\frac{5}{2}\delta)t_R} \left[C^{\beta_1} c_3 + \frac{2}{5\delta} C^{\beta_1} e^{-\frac{5}{2}\delta c_3} \right] E_1, \quad 0 \leq t_R < \infty, \\ |\mathcal{F}_{jk}(t)| &\leq \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} e^{-|k|(\rho-2\delta)} e^{(\mu-\frac{5}{2}\delta)t_R} \left[C^{\beta_2} c_3 + \frac{2}{5\delta} C^{\beta_2} e^{-\frac{5}{2}\delta c_3} \right] E_1, \quad -\infty < t_R \leq 0, \\ \|\mathcal{F}_j\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n, \\ \left\| \frac{\partial \mathcal{F}_j}{\partial q'} \right\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{8\kappa}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n, \end{aligned}$$

where $C^{\beta_3} = \max(C^{\beta_1}, C^{\beta_2})$. Now that we have obtained estimates for $\mathcal{S}_j(q)$, $\mathcal{F}_j(q')$ and their derivatives, we can obtain estimates for $Y_j(q')$, $Y(q')$ and their derivatives. Recall $Y_j(q') = \mathcal{S}_j(q) + \mathcal{F}_j(q')$. It follows

$$\begin{aligned} \|Y_j\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{\varpi}{\Gamma\delta^{2n}} \|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n, \\ \|Y\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{\varpi}{\Gamma\delta^{2n}} \|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n, \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{\partial Y_j}{\partial q'} \right\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{\varpi}{\Gamma\delta^{2n+1}} \|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n, \\ \left\| \frac{\partial Y}{\partial q'} \right\|_{\rho-3\delta, \sigma-3\delta} &\leq \frac{\varpi}{\Gamma\delta^{2n+1}} \|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n. \end{aligned}$$

Now we apply the estimates above to the generating function $\chi = X(q') + \xi \cdot q' + \sum_i Y_i(q') p'_i$. We have

$$\left\| \frac{\partial \chi}{\partial p'} \right\|_{\rho-3\delta, \sigma-3\delta} = \|Y\|_{\rho-3\delta, \sigma-3\delta} \leq \frac{\varpi}{\Gamma\delta^{2n}} \|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n. \quad (10.26)$$

Next we must relate the constant $\|\beta_j^Q(q)\|_{\rho-2\delta}$ and E_1 . To do this we find the following estimates for the terms that make up $\|\beta_j^Q(q)\|_{\rho-2\delta}$. The first term

$$\kappa \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{\partial s_k^1}{\partial p'_j}(0) e^{ik \cdot q} = \kappa \frac{\partial}{\partial p'_j} \left(\sum_{k \in \mathbb{Z}^n \setminus 0} s_k^1(p') e^{ik \cdot q} \right) \Big|_{p'=0} = \kappa \frac{\partial}{\partial p'_j} \left(G(p', q) - \overline{G}(p') \right) \Big|_{p'=0}$$

and

$$\left| \kappa \frac{\partial(G - \overline{G})}{\partial p'_j}(0, q) \right| \leq \frac{\kappa}{\delta} \|G(0, q) - \overline{G}(0)\|_{\rho-\delta} \leq \frac{2\kappa}{\delta} E_1, \quad \forall q \in \mathcal{D}_{\rho-2\delta}.$$

The second term

$$\sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) e^{ik \cdot q} = \sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \sum_{k \in \mathbb{Z}^n \setminus 0} i k_l y_k e^{ik \cdot q} = \sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \frac{\partial \mathcal{Y}}{\partial q'_l}(q),$$

and

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}^n \setminus 0} \sum_{l=1}^n i k_l y_k \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j}(0) e^{ik \cdot q} \right| &\leq \left| \sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \right| \frac{1}{\delta} \|\mathcal{Y}(q)\|_{\rho-\delta} \leq \left| \sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \right| \frac{\varpi \kappa}{\Gamma \delta^{2n+1}} \|G(0, q) - \overline{G}(0)\|_{\rho} \\ &\leq \frac{2\varpi \kappa E_1}{\Gamma \delta^{2n+1}} \left| \sum_{l=1}^n \frac{\partial^2 \tilde{H}^0}{\partial p'_l \partial p'_j} \right| \leq \frac{2\varpi \kappa E_1}{\Gamma \delta^{2n+1}} \frac{n}{\delta}, \end{aligned}$$

where we have used in the last inequality Cauchy's inequality and the fact that $\|H\|_{\rho, \sigma} \leq 1$. The third term

$$\begin{aligned} \kappa &\sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) k_l y_k e^{im \cdot q} e^{ik \cdot q} - i \frac{\partial^2 s_{-k}^1}{\partial p'_l \partial p'_j}(0) k_l y_k \right) \\ &= \kappa \sum_{l=1}^n \left(\sum_{k \in \mathbb{Z}^n} i k_l y_k e^{ik \cdot q} \right) \left(\sum_{m \in \mathbb{Z}^n} \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) e^{im \cdot q} \right) - i \kappa \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \frac{\partial^2 s_{-k}^1}{\partial p'_l \partial p'_j}(0) k_l y_k \\ &= \kappa \sum_{l=1}^n \frac{\partial \mathcal{Y}}{\partial q'_l}(q) \frac{\partial^2 G(p', q)}{\partial p'_l \partial p'_j} \Big|_{p'=0} - i \kappa \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \frac{\partial^2 s_{-k}^1}{\partial p'_l \partial p'_j}(0) k_l y_k, \end{aligned}$$

so

$$\begin{aligned} &\left| \kappa \sum_{k \in \mathbb{Z}^n} \sum_{l=1}^n \left(\sum_{m \in \mathbb{Z}^n} i \frac{\partial^2 s_m^1}{\partial p'_l \partial p'_j}(0) k_l y_k e^{im \cdot q} e^{ik \cdot q} - i \frac{\partial^2 s_{-k}^1}{\partial p'_l \partial p'_j}(0) k_l y_k \right) \right| \\ &\leq \frac{n\kappa}{\delta^2} \|\mathcal{Y}(q)\|_{\rho-\delta} \|G(0, q)\|_{\rho-\delta} + \kappa \sum_{k \in \mathbb{Z}^n} \frac{1}{\delta} \|s_{-k}^1(p')\|_{\rho-\delta} |k| |y_k| \\ &\leq \frac{n\kappa}{\delta^2} \frac{\varpi \kappa}{\Gamma \delta^{2n}} \|G(0, q) - \overline{G}(0)\|_{\rho} \|G(0, q)\|_{\rho-\delta} \\ &\quad + \kappa \sum_{k \in \mathbb{Z}^n} \frac{1}{\delta} \|G\|_{\rho-\delta} e^{-|k|(\rho-\delta)} \left(\frac{1}{e\delta} \right) e^{|k|\delta} \frac{\kappa}{\Gamma} \left(\frac{n}{e\delta} \right)^n \|G(0, q) - \overline{G}(0)\|_{\rho} e^{-|k|(\rho-\delta)} \\ &\leq \frac{2n\varpi \kappa^2}{\Gamma \delta^{2n+2}} E_1^2 + \left(\frac{\kappa^2}{e\delta^2 \Gamma} \right) \left(\frac{n}{e\delta} \right)^n E_1^2 \sum_{k \in \mathbb{Z}^n} e^{-|k|(2\rho-\delta)} \\ &= \frac{2n\varpi \kappa^2}{\Gamma \delta^{2n+2}} E_1^2 + \left(\frac{2\kappa^2}{e\delta^2 \Gamma} \right) \left(\frac{n}{e\delta} \right)^n E_1^2 \left(\frac{1}{1 - e^{-(2\rho-\delta)}} \right). \end{aligned}$$

The fourth term

$$\begin{aligned} &\left| \sum_{k \in \mathbb{Z}^n \setminus 0} \left(\sum_{l=1}^{n+1} \kappa \xi_l \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) \right) e^{ik \cdot q} \right| \leq \kappa \sum_{l=1}^{n+1} |\xi_l| \left| \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{\partial^2 s_k^1}{\partial p'_l \partial p'_j}(0) e^{ik \cdot q} \right| \\ &\leq \kappa \sum_{l=1}^{n+1} |\xi_l| \left| \frac{\partial^2}{\partial p'_l \partial p'_j} (G(p', q) - \overline{G}(p')) \Big|_{p'=0} \right| \leq \kappa \sum_{l=1}^{n+1} |\xi_l| \frac{1}{\delta} \|G(0, q) - \overline{G}(0)\|_{\rho} \leq \frac{2\kappa E_1}{\delta} |\xi|, \end{aligned}$$

and with the four estimates above we obtain

$$\|\beta_j^Q(q)\|_\rho \leq \frac{2\kappa}{\delta}E_1 + \frac{2\varpi\kappa E_1}{\Gamma\delta^{2n+1}}\frac{n}{\delta} + \frac{2n\varpi\kappa^2}{\Gamma\delta^{2n+2}}E_1^2 + \left(\frac{2\kappa^2}{e\delta^2\Gamma}\right)\left(\frac{n}{e\delta}\right)^n E_1^2 \left(\frac{1}{1-e^{-(2\rho-\delta)}}\right) + \frac{2\kappa E_1}{\delta}|\xi|.$$

Finally we have

$$\begin{aligned} \left\|\frac{\partial\chi}{\partial p'}\right\|_{\rho-3\delta,\sigma-3\delta} &\leq \frac{\varpi}{\Gamma\delta^{2n}} \left[\frac{2\kappa}{\delta}E_1 + \frac{2\varpi\kappa E_1}{\Gamma\delta^{2n+1}}\frac{n}{\delta} + \frac{2n\varpi\kappa^2}{\Gamma\delta^{2n+2}}E_1^2 + \left(\frac{2\kappa^2}{e\delta^2\Gamma}\right)\left(\frac{n}{e\delta}\right)^n E_1^2 \left(\frac{1}{1-e^{-(2\rho-\delta)}}\right) + \frac{2\kappa E_1}{\delta}|\xi| \right] \\ &\quad + \frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3}c_3 + \frac{2}{5\delta}C^{\beta_3}e^{-\frac{5}{2}\delta c_3} \right] \left(\frac{4}{\delta}\right)^n E_1. \end{aligned}$$

Next

$$\left\|\frac{\partial\chi}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta} \leq \left\|\frac{\partial X}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta} + (\rho-3\delta)(n+1) \left\|\frac{\partial Y}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta} + \|\xi\|_{\rho-3\delta,\sigma-3\delta}.$$

We put together the bounds for $\|\xi\|$, $\left\|\frac{\partial X}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta}$ and $\left\|\frac{\partial Y}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta}$ and obtain the following

$$\begin{aligned} \left\|\frac{\partial\chi}{\partial q'}\right\|_{\rho-3\delta,\sigma-3\delta} &< \frac{2\varpi\kappa}{\Gamma\delta^{2n+1}}E_1 + \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta}e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta}\right)^n E_1 \\ &\quad + (n+1)\frac{\varpi}{\Gamma\delta^{2n+1}}\|\beta_j^Q\|_{\rho-2\delta} + \frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3}c_3 + \frac{2}{5\delta}C^{\beta_3}e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta}\right)^n \\ &\quad + \frac{1}{f} \left[2\kappa E_1 + \frac{2\varpi\kappa}{m\Gamma\delta^{2n+1}}E_1 + \frac{8\kappa}{m\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta}e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta}\right)^n E_1 \right]. \end{aligned}$$

We set $\tilde{\rho} \equiv \rho - 3\delta$, $\tilde{\sigma} \equiv \sigma - 3\delta$ and examine the estimates for $\left\|\frac{\partial\chi}{\partial p'}\right\|_{\tilde{\rho},\tilde{\sigma}}$ and $\left\|\frac{\partial\chi}{\partial q'}\right\|_{\tilde{\rho},\tilde{\sigma}}$. Since $\delta < 1$, the first term of $\left\|\frac{\partial\chi}{\partial p'}\right\|_{\tilde{\rho},\tilde{\sigma}}$, from (10.26), is less than the third term of $\left\|\frac{\partial\chi}{\partial q'}\right\|_{\tilde{\rho},\tilde{\sigma}}$

$$\frac{\varpi}{\Gamma\delta^{2n}}\|\beta_j^Q(q)\|_{\rho-2\delta} < (n+1)\frac{\varpi}{\Gamma\delta^{2n+1}}\|\beta_j^Q(q)\|_{\rho-2\delta}.$$

Also, the second term of $\left\|\frac{\partial\chi}{\partial p'}\right\|_{\tilde{\rho},\tilde{\sigma}}$, from (10.26), is less than the fourth term of $\left\|\frac{\partial\chi}{\partial q'}\right\|_{\tilde{\rho},\tilde{\sigma}}$

$$\frac{8\kappa}{5\delta\gamma\sqrt{2\pi}} \left[C^{\beta_3}c_3 + \frac{2}{5\delta}C^{\beta_3}e^{-\frac{5}{2}\delta c_3} \right] E_1 < \frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3}c_3 + \frac{2}{5\delta}C^{\beta_3}e^{-\frac{5}{2}\delta c_3} \right] E_1.$$

Consequently we obtain the following estimate

$$\begin{aligned} \chi_{\tilde{\rho},\tilde{\sigma}}^* &= \max \left(\left\|\frac{\partial\chi}{\partial q'}\right\|_{\tilde{\rho},\tilde{\sigma}}, \left\|\frac{\partial\chi}{\partial p'}\right\|_{\tilde{\rho},\tilde{\sigma}} \right) = \left\|\frac{\partial\chi}{\partial q'}\right\|_{\tilde{\rho},\tilde{\sigma}} = \frac{2\varpi\kappa}{\Gamma\delta^{2n+1}}E_1 + \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta}e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta}\right)^n E_1 \\ &\quad + (n+1)\frac{\varpi}{\Gamma\delta^{2n+1}}\|\beta_j^Q(q)\|_{\rho-2\delta} + \frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3}c_3 + \frac{2}{5\delta}C^{\beta_3}e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta}\right)^n \\ &\quad + \frac{1}{f} \left[2\kappa E_1 + \frac{2\varpi\kappa}{m\Gamma\delta^{2n+1}}E_1 + \frac{8\kappa}{m\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta}e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta}\right)^n E_1 \right] \\ &\leq \frac{2\varpi\kappa}{\Gamma\delta^{2n+1}}E_1 + \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta}e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta}\right)^n E_1 + (n+1)\frac{\varpi}{\Gamma\delta^{2n+1}} \left[\frac{2\kappa}{\delta}E_1 + \frac{2\varpi\kappa E_1}{\Gamma\delta^{2n+1}}\frac{n}{\delta} + \frac{2n\varpi\kappa^2}{\Gamma\delta^{2n+2}}E_1^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2\kappa^2}{e\delta^2\Gamma} \right) \left(\frac{n}{e\delta} \right)^n E_1^2 \left(\frac{1}{1 - e^{-(2\rho-\delta)}} \right) + \frac{2\kappa E_1}{\delta} |\xi| \Big] + \frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n \\
& + \frac{1}{f} \left[2\kappa E_1 + \frac{2\varpi\kappa}{m\Gamma\delta^{2n+1}} E_1 + \frac{8\kappa}{m\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1 \right] \\
\leq & \frac{2\varpi\kappa}{\Gamma\delta^{2n+1}} E_1 + \frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1 + (n+1) \frac{\varpi}{\Gamma\delta^{2n+1}} \left[\frac{2\kappa}{\delta} E_1 + \frac{2\varpi\kappa E_1}{\Gamma\delta^{2n+1}} \frac{n}{\delta} + \frac{2n\varpi\kappa^2}{\Gamma\delta^{2n+2}} E_1^2 \right. \\
& + \left. \left(\frac{2\kappa^2}{e\delta^2\Gamma} \right) \left(\frac{n}{e\delta} \right)^n E_1^2 \left(\frac{1}{1 - e^{-(2\rho-\delta)}} \right) \right] \\
& + \frac{2\varpi\kappa(n+1)^2}{\Gamma\delta^{2n+2}f} E_1 \left[2\kappa E_1 + \frac{2\varpi\kappa}{m\Gamma\delta^{2n+1}} E_1 + \frac{8\kappa}{m\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1 \right] \\
& + \frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] E_1 \left(\frac{4}{\delta} \right)^n + \frac{1}{f} \left[2\kappa E_1 + \frac{2\varpi\kappa E_1}{m\Gamma\delta^{2n+1}} + \frac{8\kappa E_1}{m\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n \right].
\end{aligned}$$

There are thirteen terms in the estimate above which we would like to reduce to one term. We first list all the terms

- 1) $\frac{2\varpi\kappa}{\Gamma\delta^{2n+1}} E_1,$
- 2) $\frac{8\kappa}{\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1,$
- 3) $\frac{2\varpi\kappa}{\Gamma\delta^{2n+2}} (n+1) E_1,$
- 4) $\frac{2\varpi^2\kappa}{\Gamma^2\delta^{4n+3}} n(n+1) E_1,$
- 5) $\frac{2\varpi^2\kappa^2}{\Gamma^2\delta^{4n+3}} n(n+1) E_1^2,$
- 6) $\frac{2\varpi\kappa^2}{\Gamma^2 e\delta^{3n+3}} \left(\frac{n}{e} \right)^n (n+1) \left(\frac{1}{1 - e^{-(2\rho-\delta)}} \right) E_1^2,$
- 7) $\frac{4\varpi\kappa^2}{\delta^{2n+2}\Gamma f} (n+1)^2 E_1^2,$
- 8) $\frac{4\varpi^2\kappa^2}{m\Gamma^2\delta^{4n+3}f} (n+1)^2 E_1^2,$
- 9) $\frac{16\varpi\kappa^2}{m\delta^{2n+4}\Gamma f\gamma\sqrt{2\pi}} (n+1)^2 \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1^2,$
- 10) $\frac{8\kappa(n+1)}{5\delta^2\gamma\sqrt{2\pi}} \left[C^{\beta_3} c_3 + \frac{2}{5\delta} C^{\beta_3} e^{-\frac{5}{2}\delta c_3} \right] \left(\frac{4}{\delta} \right)^n E_1,$
- 11) $\frac{2\kappa}{f} E_1,$
- 12) $\frac{2\varpi\kappa}{m\Gamma f\delta^{2n+1}} E_1,$

$$13) \quad \frac{8\kappa}{fm\delta^2\gamma\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1.$$

We begin to combine terms keeping in mind $\Gamma, \gamma, \delta, m, f, \kappa E_1 < 1$ and $\varpi \gg 1$. The sum of terms 1 and 3 is bounded by

$$\frac{2\varpi\kappa}{\Gamma\delta^{2n+2}}(n+2)E_1,$$

the sum of terms 2 and 13 is bounded by

$$\frac{16\kappa}{fm\gamma\delta^2\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n E_1$$

the sum terms 1, 3, and 4, knowing that $\varpi > 1$ and assuming without loss of generality that $\Gamma < 1$, is bounded by

$$\frac{2\varpi^2\kappa}{\Gamma^2\delta^{4n+3}}(n^2 + 2n + 2)E_1,$$

the sum of terms 11 and 12 is bounded by

$$\frac{4\varpi\kappa}{m\Gamma f\delta^{2n+1}}E_1,$$

the sum of terms 1, 3, 4, 11, and 12 is bounded by

$$\frac{2\varpi^2\kappa}{mf\Gamma^2\delta^{4n+3}}(n^2 + 2n + 4)E_1,$$

the sum of terms 7 and 8 is bounded by

$$\frac{8\varpi\kappa^2}{m\Gamma^2 f\delta^{4n+3}}(n+1)^2 E_1^2,$$

the sum of terms 7, 8, and 5 is bounded by

$$\frac{\varpi^2\kappa^2}{mf\Gamma^2\delta^{4n+3}}(10n^2 + 18n + 8)E_1^2,$$

the sum of terms 7, 8, 5, and 6 is bounded by

$$\left(1 + \frac{1}{1 - e^{-(2\rho-\delta)}} \right) \frac{\varpi^2\kappa^2}{mf\Gamma^2\delta^{4n+3}}(10n^2 + 18n + 8)E_1^2,$$

where, to bound 6, we have used the fact $\frac{2}{e} \left(\frac{n}{e} \right)^n < \varpi$. The sum of terms 1, 3, 4, 11, 12, 7, 8, 5, and 6 is bounded by

$$\left(2 + \frac{1}{1 - e^{-(2\rho-\delta)}} \right) \frac{\varpi^2}{mf\Gamma^2\delta^{4n+3}}(10n^2 + 18n + 8)\kappa E_1, \quad (10.27)$$

the sum of terms 9, 2, and 13 is bounded by

$$\frac{16\varpi(n^2 + 2n + 2)}{fm\Gamma\gamma\delta^{2n+4}\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n \kappa E_1,$$

and the sum of terms 9, 2, 13, and 10 is bounded by

$$(1 + C^{\beta_3}) \frac{\varpi 16(n^2 + 2n + 2)}{fm\Gamma\gamma\delta^{2n+4}\sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2}c_3} \right] \left(\frac{4}{\delta} \right)^n \kappa E_1. \quad (10.28)$$

We use the fact

$$c_3 + \frac{2}{\delta} e^{-\frac{\delta}{2} c_3} < \frac{2}{\delta} (c_3 + 1),$$

together with

$$\frac{1}{1 - e^{-(2\rho - \delta)}} < \frac{1}{2\rho - \delta} + 1 \leq \frac{1}{\rho - \delta} + 1 \leq \frac{1}{\rho_*} + 1 < \frac{2}{\rho_*}$$

and combining (10.27) and (10.28) we obtain

$$\chi_{\tilde{\rho}, \tilde{\sigma}}^* < \left(\frac{2}{\rho_*} + C^{\beta_3} \right) \frac{16\varpi^2(10n^2 + 18n + 8)}{fm\Gamma^2\gamma\delta^{4n+3}} \frac{2}{\delta} (c_3 + 1) \left(\frac{4}{\delta} \right)^n \kappa E_1.$$

Recall

$$C^{\beta_1} = \frac{6^n 32(n+1)\varpi}{\delta^{5n+4}\Gamma\gamma fm} (c_3 + 1)e^\nu, \quad C^{\beta_2} = \frac{6^n 32(n+1)\varpi}{\delta^{5n+4}\Gamma\gamma fm} (c_3 + 1)e^\mu.$$

Then

$$C^{\beta_3} = \max(C^{\beta_1}, C^{\beta_2}) < \frac{6^n 32(n+1)\varpi}{\delta^{5n+4}\Gamma\gamma fm} (c_3 + 1)e^{\nu+\mu}.$$

Finally

$$\chi_{\tilde{\rho}, \tilde{\sigma}}^* < \frac{3^n 2^{12} \varpi^3 (6n+6)^3 (c_3+1)^2}{\rho_* f^2 m^2 \Gamma^3 \gamma^2 \delta^{9n+8}} e^{\nu+\mu} \left(\frac{4}{\delta} \right)^n \kappa E_1 = \frac{\delta}{6n+6} \eta,$$

where

$$\eta = \frac{\Lambda}{f^2 m^2 \delta^{\aleph}} \kappa E_1, \tag{10.29}$$

$$\aleph = 10n + 9, \tag{10.30}$$

$$\Lambda = \frac{3^n 2^{2n+12} \varpi^3 (6n+6)^4 (c_3+1)^2}{\rho_* \Gamma^3 \gamma^2} e^{\nu+\mu}. \tag{10.31}$$

With the estimates on the derivatives of the generating function one can now estimate the analyticity domain of the flow associated with the corresponding Hamiltonian vector field. We have obtain the following

$$\chi_{\tilde{\rho}, \tilde{\sigma}}^* < \frac{\delta}{6n+6} \eta \tag{10.32}$$

where

$$\tilde{\rho} = \rho - 3\delta, \quad \tilde{\sigma} = \sigma - 3\delta.$$

In particular, note by (10.1) it follows $\eta < 1$ since it can be written in the form

$$\eta < \sigma_*^2 \frac{m - m_*}{4(n+1)},$$

and therefore

$$\chi_{\tilde{\rho}, \tilde{\sigma}}^* \leq \frac{\delta}{6n+6} \eta \leq \frac{\delta}{2}.$$

Consequently, Lemma C.2 applies and χ generates a canonical transformation

$$\phi : D_{\rho', \sigma'} \rightarrow D_{\rho, \sigma},$$

where $\rho' = \tilde{\rho} - \delta = \rho - 4\delta$, and $\sigma' = \tilde{\sigma} - \delta = \sigma - 4\delta$. We have seen $X \in \mathcal{A}_{\rho-3\delta, \sigma-3\delta}$, $Y_i \in \mathcal{A}_{\rho-3\delta, \sigma-3\delta}$, for $i = 1, \dots, n+1$ and since $\chi = X(q') + \xi \cdot q' + \sum_i Y_i(q') p'_i$ it follows $\chi \in \mathcal{A}_{\rho-3\delta, \sigma-3\delta}$. Next, we consider the following estimates.

$$\|H'\|_{\rho', \sigma'} < 1$$

From the second inequality of Lemma C.2 it follows $\|H'\|_{\rho', \sigma'} \leq \|H\|_{\rho, \sigma} < 1$.

$$F \in A_{\rho, \sigma}, \quad \|\mathcal{U}F - F\|_{\rho', \sigma'} \leq \eta \|F\|_{\rho, \sigma}$$

From Lemma C.2 we have the following estimates

$$\|\mathcal{U}F - F\|_{\tilde{\rho}-\delta, \tilde{\sigma}-\delta} \leq 4(n+1) \frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \|F\|_{\tilde{\rho}, \tilde{\sigma}}, \quad \|F\|_{\tilde{\rho}, \tilde{\sigma}} \leq \|F\|_{\rho, \sigma},$$

Using (10.32) we have

$$4(n+1) \frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \leq \left(\frac{4n+4}{6n+6} \right) \eta \leq \eta,$$

and combining these estimates we obtain

$$\|\mathcal{U}F - F\|_{\rho', \sigma'} \leq \eta \|F\|_{\tilde{\rho}, \tilde{\sigma}} \leq \eta \|F\|_{\rho, \sigma}. \quad (10.33)$$

$$m' \equiv m - \frac{4(n+1)\eta}{\sigma_*^2}$$

The approach is to construct m' so the following inequality hold $\|C'v\|_{\rho, \sigma} \leq m'^{-1}\|v\|$, $\forall v \in \mathbb{C}^n$. By definition it follows

$$C'_{i,j}(q') - C_{i,j}(q') = \frac{\partial^2}{\partial p'_i \partial p'_j} [\mathcal{U}H - H](0, q').$$

Applying Cauchy's inequality, (10.33) and the fact that $\sigma' > \sigma_*$ we obtain

$$\|C'_{i,j} - C_{i,j}\| \leq \frac{4\|\mathcal{U}H - H\|_{\rho', \sigma'}}{\sigma'^2} \leq \frac{4\eta}{\sigma_*^2},$$

from which it follows

$$\|(C' - C)v\|_{\rho', \sigma'} \leq \frac{4(n+1)\eta}{\sigma_*^2} \|v\|. \quad (10.34)$$

Using $C' = C + (C' - C)$ we have

$$\|C'v\|_{\rho', \sigma'} \leq \|Cv\|_{\rho', \sigma'} + \|(C' - C)v\|_{\rho', \sigma'}. \quad (10.35)$$

Using the estimate $\|Cv\|_{\rho', \sigma'} \leq \|Cv\|_{\rho, \sigma} \leq m^{-1}\|v\|$, along with (10.34) and (10.35) gives

$$\|C'v\|_{\rho', \sigma'} \leq (m^{-1} + \frac{4(n+1)\eta}{\sigma_*^2}) \|v\|.$$

Applying the inequality $a^{-1} + b < (a - b)^{-1}$ for $0 < b < a < 1$, the estimate becomes

$$\|C'v\|_{\rho', \sigma'} \leq \left(m - \frac{4(n+1)\eta}{\sigma_*^2} \right)^{-1} \|v\|.$$

This estimate provides the value for m'

$$m' = m - \frac{4(n+1)\eta}{\sigma_*^2}.$$

$\boxed{f'}$

We begin with the known condition

$$\left\| \begin{pmatrix} \overline{\overline{C}} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\| \geq f \|v\|. \quad (10.36)$$

Furthermore we have by definition

$$C'_{i,j}(q') - C_{i,j}(q') = \frac{\partial^2}{\partial p'_i \partial p'_j} [\mathcal{U}H - H](0, q').$$

Applying Cauchy's inequality, using (10.33) and the fact $\sigma' > \sigma_*$, gives

$$\left\| \overline{\overline{C'}}_{i,j} - \overline{\overline{C}}_{i,j} \right\| \leq \frac{\|\mathcal{U}H - H\|_{\rho', \sigma'}}{\sigma'^2} \leq \frac{\eta}{\sigma_*^2} \|H\|_{\rho, \sigma} \leq \frac{\eta}{\sigma_*^2},$$

from which it follows

$$\left\| (\overline{\overline{C'}} - \overline{\overline{C}}) v \right\| \leq \frac{(n+1)\eta}{\sigma_*^2} \|v\|. \quad (10.37)$$

Also one has $\lambda' = (1 + \kappa E_1 \zeta_A) \lambda$. From this follows $\lambda' - \lambda = \kappa E_1 \zeta_A \lambda$, and we obtain the estimate $\|(\lambda' - \lambda)v\| \leq \kappa E_1 \|\zeta_A\| \|v\|$. We now find an estimate for ζ_A . We first find a bound for

$$\begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix}.$$

We have

$$\begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} = \begin{pmatrix} \overline{\overline{C}} & \lambda^T \\ \lambda & 0 \end{pmatrix}^{-1} \begin{pmatrix} \kappa \overline{\overline{B}} - \overline{\overline{C}} \cdot \overline{\overline{\partial_{q'} X}} \\ \kappa \overline{\overline{A}} \end{pmatrix},$$

and

$$\begin{aligned} \left\| \begin{pmatrix} \kappa \overline{\overline{B}} - \overline{\overline{C}} \cdot \overline{\overline{\partial_{q'} X}} \\ \kappa \overline{\overline{A}} \end{pmatrix} \right\| &\leq \kappa \|B\|_{\rho, \sigma} + \|C \cdot \partial_{q'} X\|_{\rho-2\delta, \sigma-2\delta} + \kappa \|A\|_{\rho, \sigma} \\ &\leq 2\kappa E_1 + m^{-1} \|\partial_{q'} X\|_{\rho-2\delta, \sigma-2\delta} \\ &\leq 2\kappa E_1 + \frac{\varpi \kappa}{m \Gamma \delta^{2n+1}} \|(G - \overline{\overline{G}})(0)\|_{\rho} + \frac{8\kappa}{m \delta^2 \gamma \sqrt{2\pi}} \left[C_3 c_3 + \frac{2}{\delta} C_3 e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \\ &\leq 2\kappa E_1 + \frac{\varpi \kappa E_1}{m \Gamma \delta^{2n+1}} + \frac{8\kappa}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n E_1. \end{aligned}$$

We know

$$\left\| \begin{pmatrix} \overline{\overline{C}} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\| \geq f \|v\|.$$

Its not hard to show

$$\left\| \begin{pmatrix} \overline{\overline{C}} & \lambda^T \\ \lambda & 0 \end{pmatrix}^{-1} v \right\| \leq \frac{1}{f} \|v\|.$$

Therefore

$$\kappa E_1 \|\zeta_A\| \leq \left\| \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} \right\| \leq \frac{1}{f} \left[2 + \frac{\varpi}{m \Gamma \delta^{2n+1}} + \frac{8}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \right] \kappa E_1$$

$$\leq \frac{4^n 11 \varpi}{m f \delta^{9n+10} \Gamma \gamma} (c_3 + 1) \kappa E_1 = \Upsilon \kappa E_1. \quad (10.38)$$

Using (10.37) and (10.38) we obtain the following

$$\begin{aligned} & \left\| \begin{pmatrix} \overline{C} - \overline{C'} & \lambda^T - \lambda'^T \\ \lambda - \lambda' & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} (\overline{C} - \overline{C'}) v_1 + (\lambda^T - \lambda'^T) v_2 \\ (\lambda - \lambda') v_2 \end{pmatrix} \right\| \\ & \leq \left\| (\overline{C} - \overline{C'}) v_1 \right\| + \|(\lambda^T - \lambda'^T) v_2\| + \|(\lambda - \lambda') v_2\| \\ & \leq \frac{(n+1)\eta}{\sigma_*^2} \|v_1\| + \frac{2}{f} \left[2 + \frac{\varpi}{m \Gamma \delta^{2n+1}} + \frac{8}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \right] \kappa E_1 \|v_2\| \\ & \leq \frac{(n+1)\eta}{\sigma_*^2} \|v\| + \frac{4}{f} \kappa E_1 \|v\| + \frac{2\varpi \kappa E_1}{f m \Gamma \delta^{2n+1}} \|v\| + \frac{16}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \kappa E_1 \|v\|, \quad (10.39) \end{aligned}$$

where $v = (v_1, v_2)$ and clearly $\|v_1\| \leq \|v\|$ and $\|v_2\| \leq \|v\|$. Using (10.39) and (10.36) we obtain

$$\begin{aligned} & \left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\| - \left\| \begin{pmatrix} \overline{C} - \overline{C'} & \lambda^T - \lambda'^T \\ \lambda - \lambda' & 0 \end{pmatrix} v \right\| \\ & \geq \left[f - \frac{(n+1)\eta}{\sigma_*^2} - \left(\frac{4}{f} + \frac{2\varpi}{f m \Gamma \delta^{2n+1}} \right) \kappa E_1 - \frac{16 \kappa E_1}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \right] \|v\|. \quad (10.40) \end{aligned}$$

For any matrix M one can write $M' = M + (M' - M)$ and $\|M'\| \geq \|M\| - \|M' - M\|$. Together with (10.40) we can bound from below as follows

$$\begin{aligned} \left\| \begin{pmatrix} \overline{C'} & \lambda'^T \\ \lambda' & 0 \end{pmatrix} v \right\| & \geq \left[f - \frac{(n+1)\eta}{\sigma_*^2} - \left(\frac{4}{f} + \frac{2\varpi}{f m \Gamma \delta^{2n+1}} \right) \kappa E_1 - \frac{16}{m \delta^2 \gamma \sqrt{2\pi}} \left[c_3 + \frac{2}{\delta} e^{-\frac{\delta c_3}{2}} \right] \left(\frac{4}{\delta} \right)^n \kappa E_1 \right] \|v\| \\ & \geq \left(f - \frac{(n+1)\eta}{\sigma_*^2} - \eta - \eta \right) \|v\| \geq \left(f - \frac{(n+3)\eta}{\sigma_*^2} \right) \|v\| = f' \|v\|. \end{aligned}$$

With the assumption $\max(\|A\|_{\rho, \sigma}, \|B\|_{\rho, \sigma}) < E_1$, the following estimate holds

$$\|P\|_{\rho, \sigma} = \kappa \|A + B \cdot p'\|_{\rho, \sigma} < (n+2) \kappa E_1. \quad (10.41)$$

We want for the new perturbation P' a similar expression namely

$$\|P'\|_{\rho', \sigma'} = \kappa' \|A' + B' \cdot p'\|_{\rho', \sigma'} < (n+2) \kappa' E'_1. \quad (10.42)$$

$\max(\|A'\|_{\rho', \sigma'}, \|B'\|_{\rho', \sigma'}) \leq E'_1$

To arrive at this estimate we must first find estimates for $\kappa' \|A'\|_{\rho', \sigma'}$, $\kappa' \|B'\|_{\rho', \sigma'}$ and identify the values for κ' and E'_1 . We know

$$\begin{aligned} \kappa' A'(q') &= H'(0, q') - a' = H'(0, q') - \overline{H'}(0), \\ H'(p', q') &= U + P + \{\chi, U\} + [\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}]. \end{aligned}$$

By our choice of χ we obtained

$$H'(p', q') = a + (1 + \varepsilon \zeta_A) \lambda \cdot p' + \mathcal{O}(\|p'\|^2) + [\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}],$$

$$\begin{aligned}
\kappa' A'(q') &= H'(0, q') - a' \\
&= a + \mathcal{U}H - H - \{\chi, H\}(0, q') - a - \overline{[\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}]}(0) \\
&= \left(\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\} \right)(0, q') - \left(\overline{[\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}]} \right)(0).
\end{aligned}$$

Therefore we obtain the estimate $\kappa' \|A'\|_{\rho', \sigma'} \leq 2 \|\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}\|_{\rho', \sigma'}$. Using the previously obtained estimates

$$\begin{aligned}
\|\{\chi, f\}\|_{\rho-\delta, \sigma-\delta} &\leq 2(n+1) \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right) \|f\|_{\rho, \sigma}, \\
\|\mathcal{U}f - f - \{\chi, f\}\|_{\rho-\delta, \sigma-\delta} &\leq 32(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}, \\
\|P\|_{\tilde{\rho}, \tilde{\sigma}} &\leq \kappa \|A\|_{\tilde{\rho}, \tilde{\sigma}} + (n+1)\kappa \|B\|_{\tilde{\rho}, \tilde{\sigma}} \leq (n+2)\kappa E_1, \\
\|H\|_{\tilde{\rho}, \tilde{\sigma}} &< 1, \\
\rho' &= \tilde{\rho} - \delta = \rho - 4\delta, \\
\sigma' &= \tilde{\sigma} - \delta = \sigma - 4\delta, \\
\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} &\leq \frac{\eta}{6n+6},
\end{aligned}$$

we obtain the following

$$\begin{aligned}
&\|\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}\|_{\rho', \sigma'} \leq \|\{\chi, P\}\|_{\rho', \sigma'} + \|\mathcal{U}H - H - \{\chi, H\}\|_{\rho', \sigma'} \\
&\leq \|\{\chi, P\}\|_{\tilde{\rho}-\delta, \tilde{\sigma}-\delta} + \|\mathcal{U}H - H - \{\chi, H\}\|_{\tilde{\rho}-\delta, \tilde{\sigma}-\delta} \\
&\leq 2(n+1) \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right) \|P\|_{\tilde{\rho}, \tilde{\sigma}} + 32(n+1)^2 \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right)^2 \|H\|_{\tilde{\rho}, \tilde{\sigma}} \\
&\leq 2(n+1) \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right) (n+2)\kappa E_1 + 32(n+1)^2 \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right)^2 \\
&< 2(n+2)^2 \kappa E_1 \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right) + 32(n+1)^2 \left(\frac{\chi_{\tilde{\rho}, \tilde{\sigma}}^*}{\delta} \right)^2 \\
&\leq [1 + 32(n+1)^2] \left(\frac{\eta}{6n+6} \right)^2 \leq \left[\frac{1 + 32(n+1)^2}{(6n+6)^2} \right] \eta^2 \leq \eta^2.
\end{aligned} \tag{10.43}$$

Where we have used the

$$2(n+2)^2 \kappa E_1 \leq \frac{\eta}{6n+6}.$$

We then have

$$\kappa' \|A'\|_{\rho', \sigma'} < \eta^2 = \left(\frac{\Lambda}{f^2 m^2 \delta^{\aleph}} \right)^2 \kappa^2 E_1^2. \tag{10.44}$$

Next recall

$$\kappa' B'_i(q') = \frac{\partial H'}{\partial p'_i}(0, q') - \lambda_i = \frac{\partial H'}{\partial p'_i}(0, q') - \frac{\partial \tilde{H}^0}{\partial p'_i}(0).$$

Using Cauchy's inequality we obtain

$$\begin{aligned} \kappa' \|B'\|_{\rho', \sigma'} &\leq \frac{1}{\sigma'} \|(H' - \tilde{H}^0)(0)\|_{\rho', \sigma'} = \frac{1}{\sigma'} \|(H' - a)(0)\|_{\rho', \sigma'} = \frac{1}{\sigma'} \|\{\chi, P\} + \mathcal{U}H - H - \{\chi, H\}\|_{\rho', \sigma'} \\ &< \frac{\eta^2}{\sigma'} < \frac{\eta^2}{\sigma_*} = \left(\frac{\Lambda}{f^2 m^2 \delta^{\aleph}} \right)^2 \frac{\kappa^2 E_1^2}{\sigma_*}. \end{aligned} \quad (10.45)$$

Finally with (10.44) and (10.45) we set

$$\kappa' = \frac{\Lambda}{f^2 m^2 \delta^{\aleph}} \kappa^2, \quad E'_1 = \frac{\Lambda}{f^2 m^2 \delta^{\aleph}} \frac{E_1^2}{\sigma_*},$$

and clearly $\max(\|A'\|_{\rho', \sigma'}, \|B'\|_{\rho', \sigma'}) \leq E'_1$.

Every time the iterative lemma is applied an equation of the form of (10.8) must be solved. The ζ_A that is chosen at the first step must satisfy the equation of the form of (10.8) for all other steps. Therefore at the step $k+1$ we have

$$\lambda^{k+1} = (1 + \kappa^k E_1^k \zeta_A) \lambda^k = (1 + \kappa^k E_1^k \zeta_A) (1 + \zeta_k) \lambda^0 = (1 + \zeta_{k+1}) \lambda^0. \quad (10.46)$$

Therefore we need an estimate for $1 + \zeta_{k+1}$.

$$\boxed{1 + \zeta'}$$

Assuming $\frac{1}{2} < 1 + \zeta < \frac{3}{2}$, with (10.38) and $1 + \zeta' = (1 + \kappa E_1 \zeta_A)(1 + \zeta)$ it follows

$$1 + \zeta - \kappa E_1 \zeta_A (1 + \zeta) < 1 + \zeta' < 1 + \zeta + \kappa E_1 \zeta_A (1 + \zeta).$$

As will be shown later, we can assume κ to be small enough so that

$$\frac{1}{2} < 1 + \zeta - \frac{3}{2} \kappa E_1 \Upsilon < 1 + \zeta' < 1 + \zeta + \frac{3}{2} \kappa E_1 \Upsilon < \frac{3}{2},$$

where Υ is as in (10.38).

$$\boxed{L'}$$

We know $\tilde{\lambda}' = (1 + \zeta') \tilde{\lambda}^0$. Then it follows

$$|\tilde{\lambda}'| = |(1 + \zeta') \tilde{\lambda}^0| < \frac{3}{2} L_0 = L'.$$

11 Conclusion of the Proof

We now describe the set up for several steps in the application of the iterative lemma.

$$\boxed{\text{Zeroth Step}}$$

We begin with the given positive constants

$$\Gamma, \rho_0, \sigma_0, m_0, \kappa_0 < 1, \quad \rho_* < \rho_0, \quad \sigma_* < \sigma_0 \quad m_* < m_0, \quad L_0, \quad E_1^0$$

and

$$-\frac{1}{2} < \kappa_0 E_1^0 \zeta_{A_0} < \frac{1}{2}, \quad f_0 = \left| \frac{1}{2} |2m_0 - L_0| - \frac{\kappa_0 E_1^0}{\sigma_0^2} \right|.$$

We assume a Hamiltonian of the form described above defined on D_{ρ_0, σ_0}

$$H_0(p', q') = U_0(p', q') + P_0(p', q')$$

$$U_0(p', q') = a^0 + \lambda^0 \cdot p' + \frac{1}{2} \sum C_{i,j}^0(q') p'_i p'_j + R^0(p', q')$$

$$P_0(p', q') = \kappa_0 A^0(q') + \kappa_0 \sum B_i^0(q') p'_i$$

where all the functions are in $\mathcal{A}_{\rho_0, \sigma_0}$, $\tilde{\lambda}_0 \in \Omega_\Gamma$, with $|\tilde{\lambda}_0| < L_0$ and the following bounds hold

$$f_0 \|v\| \leq \left\| \begin{pmatrix} \overline{\overline{C^0}} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\|C^0 v\|_{\rho_0, \sigma_0} < m_0^{-1} \|v\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\max(\|A^0\|_{\rho_0, \sigma_0}, \|B^0\|_{\rho_0, \sigma_0}) < E_1^0.$$

We apply the iterative lemma.

First Step

Applying the iterative lemma for any $\delta_0 > 0$ such that

$$\rho_0 - 4\delta_0 > \rho_*, \quad \sigma_0 - 4\delta_0 > \sigma_*,$$

and $\kappa_0 E_1^0$ small enough that

$$m_0 - 4(n+1)\eta_0/\sigma_*^2 > m_*, \quad f_0 - \frac{(n+3)\eta_0}{\sigma_*^2} > 0,$$

there exists a canonical transformation

$$\phi_1 : D_{\rho_0-4\delta_0, \sigma_0-4\delta_0} \rightarrow D_{\rho_0, \sigma_0}, \quad \phi_1 \in \mathcal{A}_{\rho_0-4\delta_0, \sigma_0-4\delta_0}$$

that transforms the Hamiltonian into

$$H_1 = H_0 \circ \phi_1 = U_1(p', q') + P_1(p', q'),$$

which can be decomposed as done previously

$$U_1(p', q') = a^1 + \lambda^1 \cdot p' + \frac{1}{2} \sum C_{i,j}^1(q') p'_i p'_j + R^1(p', q')$$

$$P_1(p', q') = \kappa_1 A^1(q') + \kappa_1 \sum B_i^1(q') p'_i$$

with the same Γ and with the new constants

$$\rho_1 = \rho_0 - 4\delta_0 > \rho_*, \quad \sigma_1 = \sigma_0 - 4\delta_0 > \sigma_*, \quad m_1 = m_0 - 4(n+1)\frac{\eta_0}{\sigma_*^2} > m_*, \quad f_1 = f_0 - \frac{(n+3)\eta_0}{\sigma_*^2},$$

$$L_1 = \frac{3}{2}L_0, \quad \eta_1 = \frac{\Lambda}{f_0^2 m_0^2 \delta_0^8} \kappa_0 E_1^0, \quad \kappa_1 = \frac{\Lambda}{f_0^2 m_0^2 \delta_0^8} \kappa_0^2, \quad E_1^1 = \frac{\Lambda}{f_0^2 m_0^2 \delta_0^8} \frac{(E_1^0)^2}{\sigma^*}$$

with all functions analytic on D_{ρ_1, σ_1} , and bounds

$$f_1 \|v\| \leq \left\| \begin{pmatrix} \overline{\overline{C^1}} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1}$$

$$\|C^1 v\|_{\rho_1, \sigma_1} < m_1^{-1} \|v\|, \quad \forall v \in \mathbb{C}^{n+1}$$

$$\max(\|A^1\|_{\rho_1, \sigma_1}, \|B^1\|_{\rho_1, \sigma_1}) < E_1^1.$$

From the iterative lemma we know the following estimate holds

$$1 - \Upsilon_0 \kappa_0 E_1^0 < 1 + \zeta_1 = 1 + \kappa_0 E_1^0 \zeta_{A_0} < 1 + \Upsilon_0 \kappa_0 E_1^0.$$

Later we will show that in fact

$$\frac{1}{2} < 1 - \Upsilon_0 \kappa_0 E_1^0 < 1 + \zeta_1 < 1 + \Upsilon_0 \kappa_0 E_1^0 < \frac{3}{2}. \quad (11.1)$$

Using (10.46) and (11.3) we have

$$|\tilde{\lambda}_1| = |(1 + \zeta_1)\tilde{\lambda}_0| \leq (1 + |\zeta_1|)|\tilde{\lambda}_0| < \frac{3}{2}L_0 = L_*.$$

Moreover, for $f \in \mathcal{A}_{\rho_1, \sigma_1}$, we have $\|\mathcal{U}_1 f - f\|_{\rho_1, \sigma_1} \leq \eta_0 \|f\|_{\rho_0, \sigma_0}$, where $\mathcal{U}_1 f \equiv f \circ \phi_1$.

Second Step

Applying the iterative lemma for any $\delta_1 > 0$ such that

$$\rho_1 - 4\delta_1 > \rho_*, \quad \sigma_1 - 4\delta_1 > \sigma_*,$$

and $\kappa_1 E_1^1$ so small that

$$m_1 - 4(n+1)\eta_1/\sigma_*^2 > m_*, \quad f_1 - \frac{(n+3)\eta_1}{\sigma_*^2} > 0,$$

there exists a canonical transformation

$$\phi_2 : D_{\rho_1 - 4\delta_1, \sigma_1 - 4\delta_1} \rightarrow D_{\rho_1, \sigma_1}, \quad \phi_2 \in \mathcal{A}_{\rho_1 - 4\delta_1, \sigma_1 - 4\delta_1}$$

that transforms the Hamiltonian into

$$H_2 = H_1 \circ \phi_2 = U_2(p', q') + P_2(p', q'),$$

which can be decomposed as done previously

$$U_2(p', q') = a^2 + \lambda^2 \cdot p' + \frac{1}{2} \sum C_{i,j}^2(q') p'_i p'_j + R^2(p', q')$$

$$P_2(p', q') = \kappa_2 A^2(q') + \kappa_2 \sum B_i^2(q') p'_i$$

with the same Γ and with the new constants

$$\rho_2 = \rho_1 - 4\delta_1 > \rho_*, \quad \sigma_2 = \sigma_1 - 4\delta_1 > \sigma_*, \quad m_2 = m_1 - 4(n+1)\frac{\eta_1}{\sigma_*^2} > m_*, \quad f_2 = f_1 - \frac{(n+3)\eta_1}{\sigma_*^2},$$

$$L_2 = \frac{3}{2}L_0, \quad \eta_2 = \frac{\Lambda}{f_1^2 m_1^2 \delta_1^8} \kappa_1 E_1^1, \quad \kappa_2 = \frac{\Lambda}{f_1^2 m_1^2 \delta_1^8} \kappa_1^2, \quad E_1^2 = \frac{\Lambda}{f_1^2 m_1^2 \delta_1^8} \frac{(E_1^1)^2}{\sigma^*},$$

with all functions analytic on D_{ρ_1, σ_1} , and bounds

$$f_2 \|v\| \leq \left\| \begin{pmatrix} \overline{C^2} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\|C^2 v\|_{\rho_2, \sigma_2} < m_2^{-1} \|v\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\max(\|A^2\|_{\rho_2, \sigma_2}, \|B^2\|_{\rho_2, \sigma_2}) < E_1^2.$$

From the iterative lemma we know the following estimate holds

$$1 - \kappa_0 E_1^0 \zeta_{A_0} - \frac{3}{2} \zeta_1 \kappa_1 E_1^1 < 1 + \zeta_2 < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \frac{3}{2} \zeta_1 \kappa_1 E_1^1.$$

Later we will show that in fact

$$\frac{1}{2} < 1 - \kappa_0 E_1^0 \zeta_A - \frac{3}{2} \Upsilon_1 \kappa_1 E_1^1 < 1 + \zeta_2 < 1 + \kappa_0 E_1^0 \zeta_A + \frac{3}{2} \Upsilon_1 \kappa_1 E_1^1 < \frac{3}{2}. \quad (11.2)$$

Using (10.46) and (11.3) we have

$$|\tilde{\lambda}_2| = |(1 + \zeta_2) \tilde{\lambda}_0| \leq (1 + |\zeta_2|) |\tilde{\lambda}_0| < \frac{3}{2} L_0 = L_*.$$

Moreover, for $f \in \mathcal{A}_{\rho_2, \sigma_2}$, we have $\|\mathcal{U}_2 f - f\|_{\rho_2, \sigma_2} \leq \eta_1 \|f\|_{\rho_1, \sigma_1}$, where $\mathcal{U}_2 f \equiv f \circ \phi_2$.

(k + 1)th Step.

We repeatedly apply the iterative lemma $k + 1$ times. We choose δ_k at each step small enough so that

$$\rho_k - 4\delta_k > \rho_*, \quad \sigma_k - 4\delta_k > \sigma_*,$$

and $\kappa_k E_1^k$ small enough that

$$m_k - 4(n+1)\eta_k/\sigma_*^2 > m_*, \quad f_k - \frac{(n+3)\eta_k}{\sigma_*^2} > 0.$$

There exists a canonical transformation

$$\phi_{k+1} : D_{\rho_k - 4\delta_k, \sigma_k - 4\delta_k} \rightarrow D_{\rho_k, \sigma_k}, \quad \phi_{k+1} \in \mathcal{A}_{\rho_k - 4\delta_k, \sigma_k - 4\delta_k},$$

that transforms the Hamiltonian into

$$H_{k+1} = H_k \circ \phi_{k+1} = U_{k+1}(p', q') + P_{k+1}(p', q'),$$

which can be decomposed as done previously

$$U_{k+1}(p', q') = a^{k+1} + \lambda^{k+1} \cdot p' + \frac{1}{2} \sum C_{i,j}^{k+1}(q') p'_i p'_j + R^{k+1}(p', q'),$$

$$P_{k+1}(p', q') = \kappa_{k+1} A^{k+1}(q') + \kappa_{k+1} \sum B_i^{k+1}(q') p'_i,$$

with the same Γ and with the new constants

$$\begin{aligned} \rho_{k+1} &= \rho_k - 4\delta_k > \rho_*, \quad \sigma_{k+1} = \sigma_k - 4\delta_k > \sigma_*, \quad m_{k+1} = m_k - 4(n+1) \frac{\eta_k}{\sigma_*^2} > m_*, \\ f_{k+1} &= f_k - \frac{(n+3)\eta_k}{\sigma_*^2}, \quad L_{k+1} = \frac{3}{2} L_0, \quad \eta_{k+1} = \frac{\Lambda}{f_k^2 m_k^2 \delta_k^8} \kappa_k E_1^k, \\ \kappa_{k+1} &= \frac{\Lambda}{f_k^2 m_0^2 \delta_k^8} \kappa_k^2, \quad E_1^{k+1} = \frac{\Lambda}{f_k^2 m_k^2 \delta_k^8} \frac{(E_1^k)^2}{\sigma_*}, \end{aligned}$$

with all functions analytic on $D_{\rho_{k+1}, \sigma_{k+1}}$, with bounds

$$\begin{aligned} f_{k+1} \|v\| &\leq \left\| \begin{pmatrix} \overline{C^{k+1}} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1}, \\ \|C^{k+1} v\|_{\rho_{k+1}, \sigma_{k+1}} &< m_{k+1}^{-1} \|v\|, \quad \forall v \in \mathbb{C}^{n+1}, \\ \max(\|A^{k+1}\|_{\rho_{k+1}, \sigma_{k+1}}, \|B^{k+1}\|_{\rho_{k+1}, \sigma_{k+1}}) &< E_1^{k+1}. \end{aligned}$$

From the iterative lemma we know the following estimate holds

$$1 - \kappa_0 E_1^0 \zeta_{A_0} - \cdots - \frac{3}{2} \zeta_{k-1} \kappa_{k-1} E_1^{k-1} - \frac{3}{2} \zeta_k \kappa_k E_1^k < 1 + \zeta_{k+1} < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \cdots + \frac{3}{2} \zeta_{k-1} \kappa_{k-1} E_1^{k-1} + \frac{3}{2} \zeta_k \kappa_k E_1^k.$$

Later we will show that in fact

$$\frac{1}{2} < 1 - \kappa_0 E_1^0 \zeta_{A_0} - \cdots - \frac{3}{2} \Upsilon_k \kappa_k E_1^k < 1 + \zeta_{k+1} < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \cdots + \frac{3}{2} \Upsilon_k \kappa_k E_1^k < \frac{3}{2}. \quad (11.3)$$

Using (10.46) and (11.3) we have

$$|\tilde{\lambda}_{k+1}| = |(1 + \zeta_{k+1}) \tilde{\lambda}_0| \leq (1 + |\zeta_{k+1}|) |\tilde{\lambda}_0| < \frac{3}{2} L_0 = L_*.$$

Moreover, for $f \in \mathcal{A}_{\rho_{k+1}, \sigma_{k+1}}$, we have $\|\mathcal{U}_{k+1} f - f\|_{\rho_{k+1}, \sigma_{k+1}} \leq \eta_k \|f\|_{\rho_k, \sigma_k}$, where $\mathcal{U}_{k+1} f \equiv f \circ \phi_{k+1}$.

Limit as $k \rightarrow \infty$

In the limit case we formally obtain the Hamiltonian

$$H_\infty(p', q') = U_\infty(p', q') + P_\infty(p', q'),$$

which can be decomposed as before

$$U_\infty(p', q') = a^\infty + \lambda^\infty \cdot p' + \frac{1}{2} \sum C_{i,j}^\infty(q') p'_i p'_j + R^\infty(p', q'),$$

$$P_\infty(p', q') = \varepsilon_\infty A^\infty(q') + \varepsilon_\infty \sum B_i^\infty(q') p'_i,$$

with constants λ^∞ , ρ_∞ , σ_∞ , m_∞ , f_∞ , L_∞ , κ_∞ , E_1^∞ where all functions are analytic on $D_{\rho_\infty, \sigma_\infty}$ with bounds

$$f_\infty \|v\| \leq \left\| \begin{pmatrix} \overline{\overline{C}}^\infty & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\|C^\infty v\|_{\rho_\infty, \sigma_\infty} < m_\infty^{-1} \|v\|, \quad \forall v \in \mathbb{C}^{n+1},$$

$$\max(\|A^\infty\|_{\rho_\infty, \sigma_\infty}, \|B^\infty\|_{\rho_\infty, \sigma_\infty}) < E_1^\infty.$$

For the expressions in the limit $k \rightarrow \infty$ to have meaning it is necessary to show

$$\rho_\infty > \rho_*, \quad \sigma_\infty > \sigma_*, \quad m_\infty > m_*, \quad L_\infty = \frac{3}{2} L_0 = L_*, \quad f_\infty > f_*$$

which will allow to specify ρ_* , σ_* and m_* , and will lead to $\kappa_\infty = 0$, i.e. the perturbation vanishes in the limit.

It is also necessary to show that the sequence of canonical transformations defined by

$$\widehat{\phi}_k : D_{\rho_k, \sigma_k} \rightarrow D_{\rho_0, \sigma_0}, \quad \widehat{\phi}_k \equiv \phi_1 \circ \dots \circ \phi_k,$$

where

$$\phi_k : D_{\rho_k, \sigma_k} \rightarrow D_{\rho_{k-1}, \sigma_{k-1}},$$

converges to an analytic, canonical transformation.

We have seen the following recursion formulas amongst the different constants

$$\begin{aligned} \rho_{k+1} &= \rho_k - 4\delta_k, \quad \sigma_{k+1} = \sigma_k - 4\delta_k, \quad m_{k+1} = m_k - 4(n+1) \frac{\eta_k}{\sigma_*^2}, \quad f_{k+1} = f_k - \frac{(n+3)\eta_k}{\sigma_*^2} \\ L_{k+1} &= \frac{3}{2} L_0, \quad \kappa_{k+1} = \frac{\Lambda}{f_k^2 m_k^2 \delta_k^8} \kappa_k^2, \quad E_1^{k+1} = \frac{\Lambda}{f_k^2 m_k^2 \delta_k^8} \frac{(E_1^k)^2}{\sigma_*}, \quad \kappa_{k+1} E_1^{k+1} = \frac{\eta_k^2}{\sigma_*}. \end{aligned} \quad (11.4)$$

We also have κ_k , E_1^k and η_k related by

$$\eta_k = \frac{\Lambda}{f_k^2 m_k^2 \delta_k^8} \kappa_k E_1^k. \quad (11.5)$$

We can use the expression for $\kappa_{k+1} E_1^{k+1}$ to eliminate η_k and thus obtain the following

$$\rho_{k+1} = \rho_k - 4\delta_k, \quad (11.6)$$

$$\sigma_{k+1} = \sigma_k - 4\delta_k, \quad (11.7)$$

$$m_{k+1} = m_k - \frac{4(n+1)}{\sigma_*} (\kappa_{k+1} E_1^{k+1})^{1/2}, \quad (11.8)$$

$$f_{k+1} = f_k - \frac{(n+3)}{\sigma_*^{3/2}} (\kappa_{k+1} E_1^{k+1})^{1/2}, \quad (11.9)$$

$$L_{k+1} = \frac{3}{2} L_0, \quad (11.10)$$

$$\kappa_{k+1} E_1^{k+1} = \frac{1}{\sigma_*} \left(\frac{\Lambda_k \kappa_k E_1^k}{f_k^2 m_k^2 \delta_k^{\aleph}} \right)^2. \quad (11.11)$$

Relations (11.6) through (11.10) can be summed to give

$$\rho_\infty = \rho_0 - 4 \sum_{k=0}^{\infty} \delta_k > \rho_*, \quad (11.12)$$

$$\sigma_\infty = \sigma_0 - 4 \sum_{k=0}^{\infty} \delta_k > \sigma_*, \quad (11.13)$$

$$L_\infty = \frac{3}{2} L_0 = L_*, \quad (11.14)$$

$$f_\infty = f_0 - \frac{(n+3)}{\sigma_*^2} \sum_{k=0}^{\infty} (\kappa_{k+1} E_1^{k+1})^{1/2} > f_*, \quad (11.15)$$

$$m_\infty = m_0 - \frac{4(n+1)}{\sigma_*^2} \sum_{k=0}^{\infty} (\kappa_{k+1} E_1^{k+1})^{1/2} > m_*. \quad (11.16)$$

Note that using (11.4) we can rewrite (11.5) as follows

$$\delta_k^{\aleph} = \frac{\Lambda^2}{m_k^4 f_k^4 \sigma_*} \frac{\kappa_k^2 (E_1^k)^2}{\kappa_{k+1} E_1^{k+1}}. \quad (11.17)$$

Defining the sequence $\{\kappa_k E_1^k\}$ serves to define the sequence $\{\delta_k\}$. Now we must make an appropriate choice of the sequence $\{\varepsilon_k\}$.

First we assume the sequence $\{\kappa_k E_1^k\}$ has the form

$$\kappa_k E_1^k = C_k \kappa_0 E_1^0, \quad k = 0, 1, 2, \dots (C_0 = 1).$$

The choice of sequence $\{\kappa_k E_1^k\}$ is important since it will result in bounds for $\varepsilon_0 E_1^0$ which is the size of the initial perturbation. We define the following series

$$\sum_{k=0}^{\infty} s_k = \sum_{k=0}^{\infty} \left(\frac{C_k^2}{C_{k+1}} \right)^{\frac{1}{2\alpha}} \equiv s, \quad (11.18)$$

$$\sum_{k=0}^{\infty} t_k = \sum_{k=0}^{\infty} C_{k+1}^{\frac{1}{2}} \equiv t. \quad (11.19)$$

With the above notation (11.12) can be rewritten as

$$\rho_0 - \rho_* > 4 \left(\frac{\Lambda^2}{m_*^4 f_*^4 \sigma_*} \right)^{\frac{1}{2\aleph}} (\kappa_0 E_1^0)^{1/2\aleph} s \quad \text{or} \quad \kappa_0 E_1^0 < \left(\frac{\rho_0 - \rho_*}{4s} \right)^{2\aleph} \frac{m_*^4 f_*^4 \sigma_*}{\Lambda^2}, \quad (11.20)$$

where we have used the fact $m_* < m_k$ (which gives a more stringent bound).

With the above notation (11.13) can be rewritten as

$$\sigma_0 - \sigma_* > 4 \left(\frac{\Lambda^2}{m_*^4 f_*^4 \sigma_*} \right)^{\frac{1}{2\aleph}} (\kappa_0 E_1^0)^{1/2\aleph} s \quad \text{or} \quad \kappa_0 E_1^0 < \left(\frac{\sigma_0 - \sigma_*}{4s} \right)^{2\aleph} \frac{m_*^4 f_*^4 \sigma_*}{\Lambda^2}. \quad (11.21)$$

Similarly (11.16) can be rewritten as

$$m_0 - m_* > \frac{4(n+1)}{\sigma_*^2} (\kappa_0 E_1^0)^{1/2} t > \frac{4(n+1)}{\sigma_0^2} (\kappa_0 E_1^0)^{1/2} t$$

or

$$\kappa_0 E_1^0 < \frac{\sigma_0^4}{16} \frac{(m_0 - m_*)^2}{(n+1)^2 t^2}, \quad (11.22)$$

and (11.15) as

$$f_0 - f_* > \frac{(n+3)}{\sigma_*^2} (\kappa_0 E_1^0)^{1/2} t > \frac{(n+3)}{\sigma_0^2} (\kappa_0 E_1^0)^{1/2} t$$

or

$$\kappa_0 E_1^0 < \frac{\sigma_0^4}{(n+3)^2} \left(\frac{f_0 - f_*}{t} \right)^2. \quad (11.23)$$

For the sequence $\{C_k\}$ we make the choice

$$C_k = 2^{-2\aleph k}, \quad (11.24)$$

so (11.18) and (11.19) become

$$s = \sum_{k=0}^{\infty} \left(\frac{C_k^2}{C_{k+1}} \right)^{\frac{1}{2\aleph}} = 2 \sum_{k=0}^{\infty} 2^{-2k} = 2 \left(\frac{1}{1 - \frac{1}{2}} \right) = 4, \quad (11.25)$$

$$t = \sum_{k=0}^{\infty} C_{k+1}^{\frac{1}{2}} = 2^{-\aleph} \left(\frac{1}{1 - 2^{-\aleph}} \right) = \frac{1}{2^{\aleph} - 1} < 1. \quad (11.26)$$

We therefore obtain four estimates for ε_0 from (11.20) (11.21), (11.22), and (11.23)

$$\kappa_0 E_1^0 < \left(\frac{\rho_0 - \rho_*}{4s} \right)^{2\aleph} \frac{m_*^4 f_*^4 \sigma_*}{\Lambda^2}, \quad (11.27)$$

$$\kappa_0 E_1^0 < \left(\frac{\sigma_0 - \sigma_*}{4s} \right)^{2\aleph} \frac{m_*^4 f_*^4 \sigma_*}{\Lambda^2}, \quad (11.28)$$

$$\kappa_0 E_1^0 < \frac{\sigma_0^4}{16} \frac{(m_0 - m_*)^2}{(n+1)^2 t^2}, \quad (11.29)$$

$$\kappa_0 E_1^0 < \frac{\sigma_0^4}{(n+3)^2} \left(\frac{f_0 - f_*}{t} \right)^2. \quad (11.30)$$

We now make the following choices

$$\rho_* = \frac{\rho_0}{2}, \quad m_* = \frac{m_0}{2}, \quad f_* = \frac{f_0}{2} \quad \text{and} \quad \sigma_* = \frac{\sigma_0}{2},$$

so (11.28) and (11.27) become

$$\kappa_0 E_1^0 < \left(\frac{\sigma_0}{32} \right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_0}{2^8 \Lambda^2}, \quad (11.31)$$

$$\kappa_0 E_1^0 < \left(\frac{\rho_0}{32} \right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_0}{2^8 \Lambda^2}. \quad (11.32)$$

Since $\sigma_0 < \rho_0$

$$\left(\frac{\sigma_0}{32}\right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_*}{2^8 \Lambda^2} < \left(\frac{\rho_0}{32}\right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_*}{2^8 \Lambda^2}.$$

Also we see

$$\frac{\sigma_0^4 (m_0 - m_*)^2}{16 (n+1)^2 t^2} = \frac{\sigma_0^4}{2^6} \frac{m_0^2}{(n+1)^2}, \quad \frac{\sigma_0^4}{(n+3)^2} \left(\frac{f_0 - f_*}{t}\right)^2 = \frac{\sigma_0^4}{4(n+3)^2} f_0^2,$$

and is not hard to see

$$\left(\frac{\sigma_0}{32}\right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_*}{2^8 \Lambda^2} < \frac{\sigma_0^4}{2^6} \frac{m_0^2}{(n+1)^2}, \quad \left(\frac{\sigma_0}{32}\right)^{2\aleph} \frac{m_0^4 f_0^4 \sigma_*}{2^8 \Lambda^2} < \frac{\sigma_0^4}{4(n+3)^2} f_0^2.$$

Therefore (11.31) gives the maximum upper bound for the size of the perturbation, $\kappa_0 E_1^0$.

Now we develop the estimates for ζ_{k+1} with the assumption at the zeroth step $-1/2 < \zeta_A < 1/2$ and then take the limit as $k \rightarrow \infty$.

Assuming at the zeroth step $-1/2 < \zeta_{A_0} = \zeta_A < 1/2$, at the first step we obtain

$$1 + \zeta_1 < 1 + \kappa_0 E_1^0 \zeta_{A_0}.$$

At the second step we obtain

$$1 + \zeta_2 < 1 + \zeta_1 + \frac{3}{2} \zeta_{A_1} \kappa_1 E_1^1 < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \frac{3}{2} \zeta_{A_1} \kappa_1 E_1^1.$$

Continuing in this manner we obtain in the $(k+1)$ step

$$1 + \zeta_{k+1} < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \frac{3}{2} \sum_{i=1}^k \zeta_{A_i} \kappa_i E_1^i.$$

One obtains a similar expression for the lower bound and combining it with the upper bound we have the following

$$1 - \kappa_0 E_1^0 \zeta_{A_0} - \frac{3}{2} \bar{\kappa} < 1 + \zeta_{k+1} < 1 + \kappa_0 E_1^0 \zeta_{A_0} + \frac{3}{2} \bar{\kappa}, \quad (11.33)$$

where $\bar{\kappa} = \sum_{i=1}^k \zeta_{A_i} \kappa_i E_1^i$.

Now we look for a condition on $\bar{\kappa}$. Using the estimate on $|\zeta_A|$ we have

$$1 + \kappa_0 E_1^0 |\zeta_{A_0}| + \frac{3}{2} |\bar{\kappa}| \leq 1 + \kappa_0 E_1^0 \Upsilon_0 + \frac{3}{2} |\bar{\kappa}| \leq 1 + \frac{3}{2} \bar{\varepsilon},$$

where $\bar{\varepsilon} = \sum_{i=0}^k \Upsilon_i \kappa_i E_1^i$. Similarly for the lower bound

$$1 - \kappa_0 E_1^0 |\zeta_{A_0}| - \frac{3}{2} |\bar{\kappa}| > 1 - \kappa_0 E_1^0 \Upsilon_0 - \frac{3}{2} |\bar{\kappa}| \geq 1 - \frac{3}{2} \bar{\varepsilon}.$$

Therefore expression (11.33) becomes

$$1 - \frac{3}{2} \bar{\varepsilon} < 1 + \zeta_{k+1} < 1 + \frac{3}{2} \bar{\varepsilon}, \quad (11.34)$$

so that

$$|\zeta_{k+1}| < \frac{3}{2} \bar{\varepsilon}. \quad (11.35)$$

We want the following to hold

$$\frac{3}{2} \bar{\varepsilon} < \frac{1}{2}. \quad (11.36)$$

Therefore we want $\bar{\varepsilon}$ to be a converging series with the above upper bound.

With the previous definitions of $\kappa_k E_1^k, C_k, \Upsilon_k$ the following estimates hold

$$\lim_{k \rightarrow \infty} \frac{3}{2} \bar{\varepsilon} = \frac{3}{2} \sum_{k=0}^{\infty} \Upsilon_k \kappa_k E_1^k \leq \frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \sum_{k=0}^{\infty} \frac{\kappa_k E_1^k}{\delta_k^8}.$$

Using (11.17) in the estimate above we obtain

$$\begin{aligned} |\zeta_{\infty}| &\leq \frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \sum_{k=0}^{\infty} \frac{m_k^2 f_k^2 \sqrt{\sigma_*}}{\Lambda} \frac{(\kappa_{k+1} E_1^{k+1})^{1/2}}{\kappa_k E_1^k} \kappa_k E_1^k \\ &\leq \frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \frac{m_0^2 f_0^2 \sqrt{\sigma_*}}{\Lambda} \sum_{k=0}^{\infty} (\kappa_{k+1} E_1^{k+1})^{1/2} \\ &= \frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \frac{m_0^2 f_0^2 \sqrt{\sigma_*}}{\Lambda} (\kappa_0 E_1^0)^{1/2} t. \end{aligned}$$

We therefore require

$$\frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \frac{m_0^2 f_0^2 \sqrt{\sigma_*}}{\Lambda} (\kappa_0 E_1^0)^{1/2} < \frac{1}{2},$$

and we obtain another restriction on the perturbation

$$\kappa_0 E_1^0 < \frac{1}{9} \left(\frac{\Gamma \gamma m_* f_*}{4^n 11 \varpi} \right)^2 \left(\frac{\Lambda}{m_0^2 f_0^2 \sqrt{\sigma_*}} \right)^2. \quad (11.37)$$

We now compare (11.37) and (11.31) and choose the smallest of these bounds to be the bound on $\kappa_0 E_1^0$. After substituting for Λ in (11.37) we obtain

$$\frac{2^{22} \varpi^4 (6n+6)^8 (c_3+1)^4 E^{2(\nu+\mu)}}{\rho_*^2 \Gamma^4 \gamma^2 (11)^2 9 \sqrt{\sigma_*}}, \quad (11.38)$$

and after substituting Λ , (10.31), in (11.31) we obtain

$$\left(\frac{\sigma_0}{32} \right)^{2(10n+9)} \frac{m_0^4 f_0^4 \sigma_*}{2^8} \frac{\rho_*^2 \Gamma^6 \gamma^4}{3^{2n} 2^{4n+24} \varpi^6 (6n+6)^8 (c_3+1)^4 e^{2(\nu+\mu)}}. \quad (11.39)$$

A simple inspection reveals (11.39) is smaller and therefore the upper bound on the perturbation $\kappa_0 E_1^0$. With (11.39) we obtain the bound

$$\begin{aligned} |\zeta_{\infty}| &\leq \frac{3}{2} \left(\frac{4^n 11 \varpi}{\Gamma \gamma m_* f_*} \right) \frac{m_0^2 f_0^2 \sqrt{\sigma_*}}{\Lambda} (\kappa_0 E_1^0)^{1/2} \\ &\leq \left(\frac{\sigma_0}{32} \right)^8 \frac{3 m_0^3 f_0^3 \sigma_* \Gamma^2 \gamma \rho_*}{3^n 2^{15} \varpi^2 (6n+6)^4 (c_3+1)^2 e^{\nu+\mu} \Lambda}. \end{aligned}$$

12 Time Aperiodic Perturbation Tending to a Quasi-periodic Time Perturbation

In this section we consider a nearly integrable system generated by a Hamiltonian consisting of an integrable part and a small perturbation. As before we consider a real valued nearly-integrable Hamiltonian in action-angle variables of the form

$$H(p, q, t) = H^0(p) + \kappa H^1(p, q, t),$$

where $p = (p_1, \dots, p_n) \in B \subset \mathbb{R}^n$, $q = (q_1, \dots, q_n) \in T^n$ are, respectively, the action and angle variables, and $\kappa \in \mathbb{R}$ is a small perturbation parameter. The form of the perturbation is as follows

$$H^1(p, q, t) = Q(p, q, t) + E(p, q, t),$$

$$Q(p, q, t) = \sum_{k \in \mathbb{Z}^{n+m}} g_k e^{ik \cdot (q, \theta t)},$$

$$E(p, q, t) = \sum_{k \in \mathbb{Z}^n} f_k(p) e_k(t) e^{ik \cdot q},$$

where $Q(p, q, t)$ is time quasi-periodic, with $\theta = (\theta_1, \dots, \theta_m)$ as the vector of basic frequencies. $f(p)$ is a bounded function and $e_k(t)$ decays exponentially to zero as time goes to infinity. Consequently, as time goes to infinity the entire perturbation tends to a t -quasi-periodic function.

The result of the theorem will be similar to the KAM-type result obtained previously in this paper. Mainly, we will prove the preservation of cylinders of the form $\mathbb{T}^n \times \mathbb{R}$ where the tori \mathbb{T}^n can be identified with the tori of the integrable system generated by $H^0(p)$. We transform the non-autonomous system into an autonomous system making each of $\theta_1 t, \dots, \theta_m t$ a dependent variable q_{n+1}, \dots, q_{n+m} respectively. We will use either notation throughout depending on need to clarify. We add conjugate variables $\tau = (\tau_1, \dots, \tau_m)$ which are needed to obtain a Hamiltonian form. We will sometimes use the notation $p' = (p, \tau)$ and $q' = (q, \theta t)$. The Hamiltonian takes the form

$$H(p, q, t) = \theta \cdot \tau + H^0(p) + \kappa H^1(p, q, t) = \tilde{H}^0(p, \tau) + \kappa H^1(p, q, t),$$

and Hamilton's equations are given by

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial H^1}{\partial q}, \\ \dot{\tau} &= -\frac{\partial H}{\partial(\theta t)} = -\frac{\partial H^1}{\partial(\theta t)}, \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{\partial H^0}{\partial p} + \frac{\partial H^1}{\partial p}, \\ (\dot{\theta t}) &= \frac{\partial H}{\partial \tau} = \theta. \end{aligned}$$

The angular frequencies are given by $\lambda(p') = \frac{\partial \tilde{H}^0}{\partial p'}(p') = (\frac{\partial H^0}{\partial p_1}, \dots, \frac{\partial H^0}{\partial p_n}, \theta_1, \dots, \theta_m)$. We will often use the notation $\lambda = (\tilde{\lambda}, \theta)$ where $\tilde{\lambda} = (\frac{\partial H^0}{\partial p_1}, \dots, \frac{\partial H^0}{\partial p_n})$. We define the complex extension of $\mathbb{R}^n \times \mathbb{R}^m \times T^n \times \mathbb{R}^m$ as follows

$$D_{\rho, \sigma, p_0} = \{(p, \tau, p, \theta t) \in \mathbb{C}^{2n+2m} \mid \|p - p_0\| \leq \rho, \|\tau\| \leq \sigma, \text{Re } q \in \mathbb{R}^n \bmod 2\pi, \|\text{Im } q\| \leq \rho, |\text{Im } t| \leq \sigma\},$$

and define $\mathcal{A}_{\rho, \sigma, p_0}$ as the set of all complex continuous functions defined on D_{ρ, σ, p_0} , analytic in the interior of D_{ρ, σ, p_0} and real for real values of the variables. The normal form is again given as the Taylor expansion of the Hamiltonian about $p' = 0$. Namely

$$H(p, \tau, q, \theta t) = a + \lambda \cdot p' + A(q') + B(q') \cdot p' + \frac{1}{2} \sum_{i,j}^{n+m} C_{i,j}(q') p'_i p'_j + R(p', q'), \quad (12.1)$$

where

$$a = \overline{\overline{H(0)}} = \overline{\tilde{H}^0(0)} + \kappa \overline{\overline{H^1(0)}} = \tilde{H}^0(0) + \kappa \overline{\overline{H^1(0)}},$$

$$\kappa A(q') = H(0, q') - a = \kappa(H^1(0, q, \theta t) - \overline{\overline{H^1(0)}}),$$

$$\begin{aligned}\kappa B_i(q') &= \frac{\partial H}{\partial p'_i}(0, q') - \lambda_i = \kappa \frac{\partial H^1}{\partial p'_i}(0, q, \theta t), \\ C_{ij}(q') &= \frac{\partial^2 H}{\partial p'_i \partial p'_j}(0, q') = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(0) + \kappa \frac{\partial^2 H^1}{\partial p'_i \partial p'_j}(0, q, \theta t),\end{aligned}$$

and

$$a \in \mathbb{R}, A, B_i, C_{i,j}, R \in A_{\rho, \sigma}, R = \mathcal{O}(\|p'^3\|).$$

The matrix \tilde{C}^* will be defined as before

$$\tilde{C}_{i,j}^* = \frac{\partial^2 \tilde{H}^0}{\partial p_j \partial p_j}(0), \quad i, j = 1, \dots, n,$$

and the same non-degeneracy condition will hold

$$\det \left(\frac{\partial^2 \tilde{H}^0}{\partial p_j \partial p_j}(0) \right) \neq 0,$$

or $d\|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\| \leq d^{-1}\|\tilde{v}\|$, $\forall \tilde{v} \in \mathbb{C}^n$, for some positive constant d . As before, this non-degeneracy condition will imply an isoenergetic non-degeneracy condition as proven in lemma 5.3. In other words, although the matrix $C_{i,j}^* = \frac{\partial^2 \tilde{H}^0}{\partial p'_j \partial p'_j}(0)$ $i, j = 1, \dots, n, n+1, \dots, n+m$ is a bigger matrix, the matrix \mathbb{I} defined as

$$\mathbb{I} = \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} \tilde{C}^* & 0 & \tilde{\lambda}^T \\ 0 & 0 & \theta^T \\ \tilde{\lambda} & \theta & 0 \end{pmatrix},$$

can be shown to be nonsingular and therefore a lower bound can be found for $\|\mathbb{I}v\|_{\rho, \sigma}$, $\forall v \in \mathbb{C}^{n+m+1}$. Using this lower bound, a similar constant, f , can be found to bound the expression

$$\left\| \begin{pmatrix} \bar{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|,$$

as was done in lemma 5.4. Ultimately we will need the implied invertibility of the matrix above to solve completely for the generating function.

Because the form of the Hamiltonian perturbation in this section is different from the one considered previously in the paper, we will re-define what we mean by p', q' -exponential form. In this section, functions of p', q' -exponential form will be understood to have the following form

$$F(p', q') = f(p') + r(p', q') + g(p', q'),$$

$$r(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k(p') e^{ik \cdot (q, \theta t)},$$

$$g(p', q') = \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

where $F(p', q') \in \mathcal{A}_{\rho, \sigma}$ and as before $e_k(t)$ is of exponential order with respect to time. Clearly the only difference in the definition is the quasi-periodic time dependence added to the term $r(p', q')$. Before examining how this modification will affect the lemmas proven in section F we prove a lemma concerning the new form of $r(p', q')$.

Lemma 12.1

Given a function $r(p', q') \in \mathcal{A}_{\rho, \sigma}$ of the form

$$r(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k(p') e^{ik \cdot (q, \theta t)},$$

it follows $\bar{\bar{r}} = s_0(p')$.

Proof: We define the vector $k = (h_1, h_2) \in \mathbb{Z}^{n+m}$ where $h_1 \in \mathbb{Z}^n$ and $h_2 \in \mathbb{Z}^m$ and

$$\begin{aligned}
\bar{\bar{r}} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{T^n} \sum_{k \in \mathbb{Z}^{n+m}} s_k(p') e^{ik \cdot (q, \theta t)} dq dt \\
&= \sum_{h_2 \in \mathbb{Z}^m} s_{(0, h_2)}(p') \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{ih_2 \cdot \theta t} dt \\
&= s_0(p') + \sum_{h_2 \in \mathbb{Z}^m \setminus 0} s_{(0, h_2)}(p') \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{ih_2 \cdot \theta t} dt \\
&= s_0(p') + \sum_{h_2 \in \mathbb{Z}^m \setminus 0} s_{(0, h_2)}(p') \lim_{T \rightarrow \infty} \frac{\sin(h_2 \cdot \theta T)}{2T} = s_0(p') \quad \square
\end{aligned}$$

The result of lemma F.1 still holds. With the result of lemma 12.1, lemma F.2 still holds. That is, given

$$F(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k(p') e^{ik \cdot (q, \theta t)} + \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

it follows

$$F(0, p') - \bar{\bar{F}} = \sum_{k \in \mathbb{Z}^{n+m} \setminus 0} s_k(0) e^{ik \cdot (q, \theta t)} + \sum_{k \in \mathbb{Z}^n} h_k(0) e_k(t) e^{ik \cdot q}.$$

The results and proofs of lemmas comparable to lemmas F.3 and F.4 do not change and follow the same arguments. A lemma comparable to lemma F.5 concerning the time derivative of a function $F(p', q')$ of p', q' -exponential form would follow the same line of arguments but one should take in consideration the time derivative of the function $r(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k(p') e^{ik \cdot (q, \theta t)}$. Since such time derivative results in a q periodic time quasi-periodic function, the total form of $\partial F(p', q') / \partial t$ will be of p', q' -exponential form and the domain of analyticity would be as stated in lemma F.5. Finally, the added time quasi-periodic dependence of $r(p', q')$ will not change the result of lemma F.7 concerning the p', q' -exponential form for the Poisson bracket $\{\chi, F\}$. The arguments of the proof would follow the same and some sums would have to change from $\sum_{j=1}^n$ to $\sum_{j=1}^{n+1}$ to account for time derivatives that in lemma F.7 are zero but now must have to be added.

The statement and proof of a lemma comparable to the iterative lemma 10.1 would follow exactly the same up to the generating function used and the definition of the perturbation (10.12). The generating function will have almost the same form

$$\chi = X(q') + \xi \cdot q' + Y(q') \cdot p',$$

$$X(q') = \mathcal{Y}(q') + \mathcal{T}(q'),$$

$$\mathcal{Y}(q') = \sum_{k \in \mathbb{Z}^{n+m}} y_k e^{ik \cdot (q, \theta t)}, \quad \mathcal{T}(q') = \sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q},$$

$$Y_j(q') = \mathcal{S}_j(q') + \mathcal{F}_j(q'),$$

$$\mathcal{S}_j(q') = \sum_{k \in \mathbb{Z}^{n+m}} \mathcal{S}_{k,j} e^{ik \cdot (q, \theta t)}, \quad \mathcal{F}_j(q') = \sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q},$$

where now $\mathcal{Y}(q')$ and $\mathcal{S}_j(q')$ depend quasi-periodically on time. The new perturbation must have the following form

$$H^1(p', q') = G(p', q') + T(p', q') \in \mathcal{A}_{p, \sigma},$$

$$G(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k^1(p') e^{ik \cdot (p, \theta t)} \in \mathcal{A}_{\rho, \sigma},$$

$$T(p', q') = \sum_{k \in \mathbb{Z}^n} h_k^1(p') e_k^1(t) e^{ik \cdot q} \in \mathcal{A}_{\rho, \sigma},$$

where $e_k^1(t)$ will be assumed to have the same properties as before. Solving for $X(q')$ involves solving the equation $\lambda \cdot \partial_{q'} X(q') = \kappa \left(A(q') - \overline{A} \right)$, which can be separated into a quasi-periodic part and an exponential-order-with-respect-to-time part. These two problems are given by

$$\lambda \cdot \partial_{q'} \mathcal{Y}(q') = \kappa \left(G(0, q') - \overline{G}(0) \right), \quad (12.2)$$

$$\lambda \cdot \partial_{q'} T(q') = \kappa T(0, q'), \quad (12.3)$$

respectively. Solving (12.2) will follow as before but now we must make use of the diophantine condition on $\lambda \in \Omega_\Gamma$ rather than just on $\tilde{\lambda}$. As before we set

$$\overline{(\kappa B - C \cdot \xi - C \cdot \partial_{q'} X)} = \kappa E_1 \zeta_A \lambda.$$

This leads to the conditions

$$\overline{C} \cdot \xi + \kappa E_1 \zeta_A \lambda = \kappa \overline{B} - \overline{C} \cdot \overline{\partial_{q'} X}, \quad \lambda \cdot \xi = \kappa \overline{A},$$

or equivalently

$$\begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \kappa E_1 \zeta_A \end{pmatrix} = \begin{pmatrix} \kappa \overline{B} - \overline{C} \cdot \overline{\partial_{q'} X} \\ \kappa \overline{A} \end{pmatrix}. \quad (12.4)$$

Since we are interested in solving the above equation for the unknowns ξ and ζ_A , we would like the matrix on the left hand side to be nonsingular. Using the fact the matrix \mathbb{I} is nonsingular together with a lemma comparable to lemma (5.4) it follows easily the matrix on the left hand side of (12.4) is nonsingular. The solution of (12.3) follows exactly as shown previously in the paper. Next we look at the solution of

$$\lambda \cdot \partial_{q'} Y = \beta, \quad (12.5)$$

where as before β is defined as

$$\beta = \left[(\kappa B(q') - C(q') \cdot \xi - C(q') \cdot \partial_{q'} X(q')) - (\kappa \overline{B} - \overline{C} \cdot \xi - \overline{C} \cdot \overline{\partial_{q'} X}) \right].$$

Since the definition of $H^1(p', q')$ has a quasi-periodic time dependence not present before, $B_j(q')$, $C(q')$ and $X(q')$ in terms of H^1 will change. Consequently the terms that make up β will change. For instance

$$\kappa \left(B_j(q') - \overline{B}_j \right) = \kappa \left[\sum_{k \in \mathbb{Z}^{n+m} \setminus 0} \frac{\partial s_k^1}{\partial p_j'}(0) e^{ik \cdot (q, \theta t)} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k^1}{\partial p_j'}(0) e_k^1(t) e^{ik \cdot q} \right].$$

Furthermore, when calculating $(C(q') \cdot \partial_{q'} X(q'))_j$ three more terms must be added to this expression

$$\frac{\partial^2 \tilde{H}^0}{\partial p_{n+1}' \partial p_j'}(0) \partial_{q_{n+1}'} \mathcal{Y}(q'), \quad \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 s_k^1}{\partial p_{n+1}' \partial p_j'}(0) e^{ik \cdot q} \partial_{q_{n+1}'} \mathcal{Y}(q'), \quad \kappa \sum_{k \in \mathbb{Z}^n} \frac{\partial^2 h_k^1}{\partial p_{n+1}' \partial p_j'}(0) e_k^1(t) e^{ik \cdot q} \partial_{q_{n+1}'} \mathcal{Y}(q').$$

Previously these terms would dropout of the expression since $\partial_{q_{n+1}'} \mathcal{Y}(q') = 0$. The solution of (12.5) splits into a quasi-periodic part and an exponential-with-respect-to-time part given by the following

$$\lambda \cdot \partial_{q'} \mathcal{S}_j(q') = \beta_j^Q(q') = \sum_{k \in \mathbb{Z}^{n+m} \setminus 0} \beta_{j,k}^Q e^{ik \cdot (q, \theta t)} \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta}, \quad (12.6)$$

$$\lambda \cdot \partial_{q'} \mathcal{F}_j(q') = \beta_j^E(q') = \sum_{k \in \mathbb{Z}^n} \beta_{j,k}^E(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-2\delta, \sigma-2\delta}, \quad (12.7)$$

where $\beta_j^Q(q')$ and $\beta_j^E(q')$ are obtained from the expressions for $\kappa B(q') - \kappa \bar{B}$, $C(q') \cdot \xi - \bar{C} \cdot \xi$, and $C(q') \cdot \partial_{q'} X(q') - \bar{C} \cdot \overline{\partial_{q'} X}$. As done previously in the paper one can solve (12.6) and (12.7) to obtain $Y_j(q')$ with all the necessary bounds. The rest of the bounds and the iterative process are obtained as before. The main result is described in the following theorem.

Theorem 12.1

Consider the Hamiltonian $H(p', q') = H^0(p) + \kappa H^1(p, q, t) + \tau = \tilde{H}^0(p') + \kappa H^1(p', q')$ of $(C_1, C_2, c_1, c_2, \nu, \mu)p', q'$ -exponential form defined on $B \times \mathbb{T}^n$ where $\tilde{H}^0(p') = H^0(p) + \tau$ and $\kappa \ll 1$ is a small perturbation parameter. Assume the perturbation has the following form

$$H^1(p', q') = Q(p', q') + E(p', q'),$$

$$Q(p', q') = \sum_{k \in \mathbb{Z}^{n+m}} s_k^1(p') e^{ik \cdot (q, \theta t)}, \quad E(p', q') = \sum_{k \in \mathbb{Z}^n} h_k^1(p') e_k^1(t) e^{ik \cdot q}.$$

Fix $p'_0 \in B$ and denote

$$\lambda = \lambda(p'_0) = \left(\frac{\partial H^0}{\partial p_1}, \dots, \frac{\partial H^0}{\partial p_n}, \theta_1, \dots, \theta_m \right) = \frac{\partial \tilde{H}^0}{\partial p'}(p'_0), \quad \tilde{C}_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p_i \partial p_j}(p'_0) \quad i, j = 1, \dots, n,$$

$$C_{ij}^* = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(p'_0) \quad i, j = 1, \dots, n+m, \quad \mathbb{I} = \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix}.$$

Assume there exists positive numbers $\Gamma, \gamma, \rho, \sigma, d, \kappa$, all less than one and L , such that $\rho > \sigma$ and

$$\lambda \in \Omega_\Gamma, \quad |\tilde{\lambda}| < L, \quad H^0, H^1 \in \mathcal{A}_{\rho, \sigma, p_0}, \quad d \|\tilde{v}\| \leq \|\tilde{C}^* \tilde{v}\| \leq d^{-1} \|\tilde{v}\|, \quad \forall \tilde{v} \in \mathbb{C}^n,$$

moreover, $\|H\|_{\rho, \sigma, p_0} < 1$, (the last condition will imply the isoenergetic condition on \mathbb{I}).

Then there exists positive numbers $E, \kappa', E', \zeta', \rho'$ and σ' , such that if $\|H^1\|_{\rho, \sigma, p_0} \leq E$ one can construct a canonical analytic change of variables

$$\begin{aligned} \phi : D_{\rho', \sigma'} &\rightarrow D_{\rho, \sigma, p_0}, \\ (\mathcal{P}', \mathcal{Q}') &\mapsto \phi(\mathcal{P}', \mathcal{Q}') = (p', q'), \end{aligned}$$

with $\phi \in \mathcal{A}_{\rho', \sigma'}$, which brings the Hamiltonian H into the form $H' = H \circ \phi$ given by

$$H'(\mathcal{P}', \mathcal{Q}') = (H \circ \phi)(\mathcal{P}', \mathcal{Q}') = a' + (1 + \kappa' E' \zeta') \lambda \cdot \mathcal{P}' + \mathcal{O}(\|\mathcal{P}'\|^2),$$

where $a', \zeta' \in \mathbb{R}$. The change of variables is near identity in the sense that $\|\phi - \text{identity}\|_{\rho', \sigma'} \rightarrow 0$ as $\|H^1\|_{\rho, \sigma, p_0} \rightarrow 0$. In the new coordinates Hamilton's Equations are given by

$$\dot{\mathcal{Q}}' = \frac{\partial H'}{\partial \mathcal{P}'}(\mathcal{P}', \mathcal{Q}') = (1 + \kappa' E' \zeta') \lambda + \mathcal{O}(\|\mathcal{P}'\|), \quad \dot{\mathcal{P}}' = -\frac{\partial H'}{\partial \mathcal{Q}'}(\mathcal{P}', \mathcal{Q}') = \mathcal{O}(\|\mathcal{P}'\|^2).$$

The solutions are given by

$$\mathcal{Q}'(t) = (1 + \kappa' E' \zeta') \lambda t + \mathcal{Q}'_0 \quad \mathcal{P}'(t) = \mathcal{P}'_0 = (\mathcal{P}_0, \tilde{\tau}_0) = 0.$$

We can write out explicitly these solutions $\mathcal{Q}'(t) = (\mathcal{Q}_1(t), \dots, \mathcal{Q}_n(t), \mathcal{T}(t)) = (1 + \kappa' E' \zeta') \lambda t + \mathcal{Q}'_0 = (1 + \kappa' E' \zeta')(\tilde{\lambda}, 1)t + \mathcal{Q}'_0$ which indicates in the new coordinate system the phase space $\mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ is foliated by invariant infinite cylinders each sustaining quasi-periodic motions identified by the frequency $\tilde{\lambda}(\mathcal{P}_0)$ and evolving in the t direction. Note under the transformation, the time variable $\mathcal{T}(t) = (1 + \kappa' E' \zeta')t$ has shifted by a constant proportional to the size of the perturbation.

13 Conclusions

In this paper we looked at time aperiodic perturbations of nearly integrable Hamiltonian systems. To prove a KAM-type theorem we used the Lie series formalism in an attempt to construct a series of generating Hamiltonians from which we found a series of near identity canonical transformations. This series of canonical transformations is used to put the original Hamiltonian in a normal form which has a smaller perturbation at each step of the iterative process. Since we considered an aperiodic time perturbation, the Fourier series methods normally used to solve for the generating function no longer apply. We therefore made use of Fourier transform techniques which allowed us to understand how the analyticity domain of the Hamiltonian changes through the iterative process and how fast the perturbation shrinks. We considered the general form of the Hamiltonian $H(p, \tau, q, t) = H^0(p, \tau) + \kappa H^1(p, \tau, q, t)$ where τ is conjugate to t . In the first case considered the perturbation decayed exponentially in time to a term depending only on angles. In the second case we added to this exponentially decaying term a time quasiperiodic term. In both cases we saw the time aperiodicity had to decay exponentially in order to have control over the analyticity of the Hamiltonian. Moreover, since the p', q' -exponential form of the perturbation is essential in solving the partial differential equations that give the generating function, it was necessary to prove the p', q' -exponential form of the perturbation is preserved after each step of the iterative process.

A Properties of Analytic Functions on $\mathbb{T}^n \times \mathbb{C}$

As mentioned before, techniques from multi-variable complex analysis are used in this paper. For example, a generating Hamiltonian with converging estimates can be obtained via Fourier methods assuming appropriate domains of analyticity for the perturbation of the Hamiltonian. Consequently we derive some important properties of analytic functions on $\mathbb{T}^n \times \mathbb{C}$. We recall a common Fourier expansion for a complex analytic function depending periodically on the variables q_1, \dots, q_n . Consider a multi-periodic function f analytic on some open strip of \mathbb{C}^n with periods $L_1, \dots, L_n > 0$, it is possible to represent f by the formula $f(q_1, \dots, q_n) = \sum_{k \in \mathbb{Z}^n} c_k e^{i(k_1 \omega_1 q_1 + \dots + k_n \omega_n q_n)}$ where the coefficients $c_k \in \mathbb{C}$ have the relation $c_k = \bar{c}_{-k}$, and $c_k = \int_0^{L_1} \frac{dq_1}{L_1} \dots \int_0^{L_n} \frac{dq_n}{L_n} e^{-i(\omega_1 k_1 q_1 + \dots + \omega_n k_n q_n)} f(q_1, \dots, q_n)$, where $\omega_i = 2\pi/L_i$, $i = 1, \dots, n$.

A function also depending aperiodically on a variable $t \in \mathbb{C}$ can be expressed similarly in a Fourier expansion. Let $F(q, t) \in \mathcal{A}_{\rho, \sigma, p_0}$ be real value, continuous, and 2π periodic in each q_i . One writes $F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{iq \cdot k}$, where $f_k(t) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} F(q, t) e^{-iq \cdot k} dq_1 \dots dq_n$. The following lemma gives a useful bound for the Fourier coefficients of a function $F(q, t)$ depending 2π periodically on q and aperiodically on t .

Lemma A.1

For $F \in \mathcal{A}_{\rho, \sigma}$ and given $F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q}$, then for every $k \in \mathbb{Z}^n$ and for all $t \in \mathbb{C}$ we have $|f_k(t)| \leq \|F(t)\|_{\rho} e^{-|k|\rho}$, where $|k| = \sum_i |k_i|$.

Proof : From the previous argument we can write

$$f_k(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(q_1, \dots, q_n, t) e^{-i \sum_j k_j q_j} dq_1 \dots dq_n.$$

Since $F \in \mathcal{A}_{\rho, \sigma}$, we can shift integration paths as follows

$$f_k(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} F(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho, t) \left\{ \prod_{j=1}^n e^{-ik_j (q_j - i \frac{k_j}{|k_j|} \rho)} \right\} dq_1 \dots dq_n.$$

Note in case $k_i = 0$ we don't shift the path in that direction.

Then the following estimate is clear

$$|f_k(t)| \leq \|F(t)\|_\rho e^{-(\sum |k_j|)\rho} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik \cdot q} dq \leq \|F(t)\|_\rho e^{-|k|\rho}.$$

(Note $\|F(t)\|_\rho$ is a function of t). \square

The following lemma is useful to construct a function in the form of a Fourier series with a specified strip of analyticity given the Fourier coefficients satisfy an exponentially decaying bound. It will become evident later how this will enable us to construct a generating function in the form of a Fourier series whose coefficients have a specified form depending directly on the perturbation of the Hamiltonian.

Lemma A.2

Suppose for a positive real constant $\rho < 1$ and every $k \in \mathbb{Z}^n$ one has

$$|f_k(t)| \leq C(t) e^{-|k|\rho}, \quad C(t) \geq 0, \quad \forall t \in \mathbb{C},$$

with $f_k(t) \in \mathcal{A}_\sigma$, $\sup_{D_\sigma}(C(t)) = C$, and consider the function

$$F(q, t) = \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q}.$$

For any positive δ with $\delta < \rho$ one has $F \in \mathcal{A}_{\rho-\delta, \sigma}$ and $\|F\|_{\rho-\delta, \sigma} \leq C \left(\frac{4}{\delta}\right)^n$.

Proof : Let $\|Im q\| \leq \rho - \delta$ then

$$\begin{aligned} \|F\|_{\rho-\delta, \sigma} &= \sup_{D_{\rho-\delta, \sigma}} \left| \sum_{k \in \mathbb{Z}^n} f_k(t) e^{ik \cdot q} \right| \leq C \sum_{k \in \mathbb{Z}^n} e^{-|k|\rho} e^{|k|(\rho-\delta)} = C \sum_{k \in \mathbb{Z}^n} e^{-|k|\delta} \\ &\leq C 2^n \sum_{\substack{k \in \mathbb{Z}^n \\ k_1 \geq 0 \dots k_n \geq 0}} e^{-\delta \sum k_j} = C 2^n \left(\sum_{k=0}^{\infty} e^{-\delta k} \right)^n = C 2^n \left(\frac{1}{1 - e^{-\delta}} \right)^n. \end{aligned}$$

For any positive $\delta < 1$, we have $\frac{\delta}{1-e^{-\delta}} < 2$ which implies $\frac{1}{1-e^{-\delta}} < \frac{2}{\delta}$, from which follows that

$$\|F\|_{\rho-\delta, \sigma} \leq C 2^n \left(\frac{2}{\delta}\right)^n = C \left(\frac{4}{\delta}\right)^n.$$

The bound holds for any finite sum of the terms $f_k(t) e^{ik \cdot q} \in \mathcal{A}_{\rho-\delta, \sigma}$, and any such finite sum is analytic in $D_{\rho-\delta, \sigma}$. Therefore, F is the limit of a uniformly convergent sequence of analytic functions and thus also analytic. \square

Lemma A.3

For any $K, s, \delta > 0$ one has the inequality $K^s \leq \left(\frac{s}{e\delta}\right)^s e^{K\delta}$.

Proof: For any $x \in \mathbb{R}$ we have $x \leq e^{x-1}$. Let $x = \frac{K\delta}{s}$, then

$$\frac{K\delta}{s} \leq e^{(\frac{K\delta}{s}-1)}$$

so that

$$K \leq \frac{s}{\delta e} e^{\frac{K\delta}{s}} \quad \text{or} \quad K^s \leq \left(\frac{s}{\delta e}\right)^s e^{K\delta} \quad \square$$

B Bounds on the Kolmogorov Normal Form in terms of the original Hamiltonian

Proof of Lemma 5.1

Clearly

$$C^* = \begin{pmatrix} \tilde{C}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Assuming $d\|\tilde{v}\| \leq \|\tilde{C}^*\tilde{v}\| \leq d^{-1}\|\tilde{v}\|$ where

$$\tilde{C}^*\tilde{v} = (\sum_j \tilde{C}_{1j}^* \tilde{v}_j, \dots, \sum_j \tilde{C}_{nj}^* \tilde{v}_j),$$

it follows

$$\|\tilde{C}^*\tilde{v}\| = \max_i \left| \sum_j \tilde{C}_{ij}^* \tilde{v}_j \right|.$$

Also, given $v = (\tilde{v}, w) = (\tilde{v}_1, \dots, \tilde{v}_n, w)$ for $\tilde{v} \in \mathbb{C}^n, w \in \mathbb{C}$, and $v \in \mathbb{C}^{n+1}$ it follows

$$C^*v = (\sum_j \tilde{C}_{1j}^* \tilde{v}_j, \dots, \sum_j \tilde{C}_{nj}^* \tilde{v}_j, 0)$$

and

$$\|\tilde{C}^*v\| = \max_i \left| \sum_j \tilde{C}_{ij}^* v_j \right|.$$

This implies $\|C^*v\| = \|\tilde{C}^*\tilde{v}\|$ and therefore $\|C^*v\| = \|\tilde{C}^*\tilde{v}\| \leq d^{-1}\|\tilde{v}\|$, $\forall v \in \mathbb{C}^{n+1}$ and $\forall \tilde{v} \in \mathbb{C}^n$.

Finally, since $\|\tilde{v}\| = \max(|\tilde{v}_1|, \dots, |\tilde{v}_n|) \leq \max(|\tilde{v}_1|, \dots, |\tilde{v}_n|, |w|) = \|v\|$, it follows $\|C^*v\| \leq d^{-1}\|v\|$. \square

We summarize some results and give some new definitions. We defined previously the $(n \times n)$ matrix $\tilde{C}^* = \partial_p^2 \tilde{H}^0(0)$ and assumed $d\|\tilde{v}\| \leq \|\tilde{C}^*\tilde{v}\| \leq d^{-1}\|\tilde{v}\|$ $\tilde{v} \in \mathbb{C}^n$. We defined the $(n+1) \times (n+1)$ matrix $C^* = \partial_{p'}^2 \tilde{H}^0(0)$ and obtained the upper bound $\|C^*v\| \leq d^{-1}\|v\|$ $v \in \mathbb{C}^{n+1}$. We defined the $(n+1) \times (n+1)$ matrix $C = \partial_{p'}^2 \tilde{H}^0(0) + \kappa \partial_{p'}^2 H^1(0, q')$. We define the $(n+2 \times n+2)$ isoenergetic matrix

$$\mathbb{I} = \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} \tilde{C}^* & 0 & (\tilde{\lambda})^T \\ 0 & 0 & 1 \\ \tilde{\lambda} & 1 & 0 \end{pmatrix}$$

where $\lambda = (\tilde{\lambda}, 1), \lambda \in \mathbb{C}^{n+1}$ and $\tilde{\lambda} \in \mathbb{C}^n$.

Proof of Lemma 5.2

1. $\max(\|A\|_{\rho, \sigma}, \|B\|_{\rho, \sigma}) \leq 2 \frac{E}{\sigma}.$

We assume for the original Hamiltonian $\|H^1\|_{\rho, \sigma} \leq E$. Recall

$$A(q') = (H^1(0, q') - \overline{\overline{H^1(0)}}), \quad B(q') = \frac{\partial H^1}{\partial p'}(0, q').$$

It follows $\|A\|_{\rho, \sigma} \leq 2\|H^1\|_{\rho, \sigma}$. Applying Cauchy's inequality and the fact $\rho > \sigma$ we obtain the bound

$$\|B\|_{\rho, \sigma} \leq \frac{1}{\sigma} \|H^1\|_{\rho, \sigma}.$$

Therefore

$$\|A\|_{\rho,\sigma} \leq 2E < \frac{2E}{\sigma} \quad (\sigma < 1), \quad \|B\|_{\rho,\sigma} \leq \frac{E}{\sigma} < \frac{2E}{\sigma}$$

from which follows

$$\max(\|A\|_{\rho,\sigma}, \|B\|_{\rho,\sigma}) < \frac{2E}{\sigma} = E_1.$$

2. $\|Cv\|_{\rho,\sigma} \leq m^{-1}\|v\|.$

By definition we have

$$C_{i,j}(q') = \frac{\partial^2 H}{\partial p'_i \partial p'_j}(0, q'), \quad C_{i,j}^*(0) = \frac{\partial^2 \tilde{H}^0}{\partial p'_i \partial p'_j}(0).$$

Therefore

$$C - C^* = \left\{ \kappa \frac{\partial^2 H^1}{\partial p'_i \partial p'_j}(0, q') \right\},$$

and it follows

$$\|(C - C^*)v\|_{\rho,\sigma} \leq (n+1)\|C - C^*\|_{\rho,\sigma}\|v\|.$$

Applying Cauchy's inequality we obtain

$$\|(C - C^*)v\|_{\rho,\sigma} \leq \frac{2(n+1)}{\sigma^2} \kappa \|H^1\|_{\rho,\sigma} \|v\| \leq \frac{2(n+1)}{\sigma^2} \kappa E \|v\|.$$

Recall from the statement of theorem 3.1

$$\kappa E = \left(\frac{\sigma}{32} \right)^{2N} \frac{d^4 f^4 \sigma_*}{2^{12} \Lambda^2}$$

One can verify (using $d < 1$)

$$2 \frac{(n+1)}{\sigma^2} \kappa E = 2 \frac{(n+1)}{\sigma^2} \left(\frac{\sigma}{32} \right)^{2N} \frac{d^4 f^4 \sigma_*}{2^{12} \Lambda^2} < \frac{d}{2},$$

from which follows

$$\|(C - C^*)v\|_{\rho,\sigma} \leq \frac{d}{2} \|v\|.$$

Now recall $\|C^*v\| \leq d^{-1}\|v\|$, $\forall v \in \mathbb{C}^{n+1}$. Writing $C = C^* + (C - C^*)$, we obtain

$$\|Cv\|_{\rho,\sigma} \leq \|C^*v\| + \|(C - C^*)v\|_{\rho,\sigma} \leq (d^{-1} + \frac{d}{2})\|v\| \leq 2d^{-1}\|v\|,$$

taking $m = \frac{d}{2}$, the result follows. \square

The next Lemma will be used in the proof of the Lemma on the Isoenergetic Nondegeneracy Condition.

Lemma B.1 *Given two positive functions $f(x)$ and $g(x)$ then*

$$\max_x (f - g) \geq \max_x (f) - \max_x (g).$$

Proof:

Let g_0 be fixed. Then clearly

$$\max_x(f - g_0) = \max_x(f) - g_0.$$

Next set $g_0 = \max(g)$. It follows

$$g_0 \geq g \quad \text{and} \quad f - g_0 \leq f - g.$$

Consequently

$$\max_x(f - g) \geq \max_x(f - g_0) = \max_x(f) - g_0 = \max_x(f) - \max_x(g).$$

□

Proof of Lemma 5.3

Let $v = (\tilde{v}, v_1, v_2)$ where $\tilde{v} \in \mathbb{C}^n, v_1, v_2 \in \mathbb{C}$. The following estimates follow

$$\begin{aligned} \left\| \begin{pmatrix} \tilde{C}^* & 0 & \tilde{\lambda}^T \\ 0 & 0 & 1 \\ \tilde{\lambda} & 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{v} \\ v_1 \\ v_2 \end{pmatrix} \right\|_{\rho, \sigma} &= \max \left(|v_2|, |\tilde{\lambda} \cdot \tilde{v} + v_1|, \left| \sum_j \tilde{C}_{i,j}^* v_j + v_2 \tilde{\lambda}_i \right| \right) \\ &\geq \max_i \left(\left| \sum_j \tilde{C}_{i,j}^* v_j + v_2 \tilde{\lambda}_i \right| \right) \geq \max_i \left(\left| \sum_j \tilde{C}_{i,j}^* v_j \right| - |v_2 \tilde{\lambda}_i| \right) \geq \left| \max_i \left(\left| \sum_j \tilde{C}_{i,j}^* v_j \right| \right) - \max_i (|v_2 \tilde{\lambda}_i|) \right| \\ &\geq \left| d \|v\| - L \|v\| \right| \geq \frac{1}{2} |d - L| \|v\| \end{aligned}$$

and we set $l = \frac{1}{2} |d - L|$. □

Proof of Lemma 5.4

Recall

$$\mathbb{I} = \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} C(q') & \lambda^T \\ \lambda & 0 \end{pmatrix} + \begin{pmatrix} -\kappa \partial_{p'}^2 H^1(0, q') & 0 \\ 0 & 0 \end{pmatrix}.$$

Then clearly

$$\begin{aligned} \left\| \begin{pmatrix} \overline{C} & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|_{\rho, \sigma} &\geq \left| \left\| \begin{pmatrix} C^* & \lambda^T \\ \lambda & 0 \end{pmatrix} v \right\|_{\rho, \sigma} - \left\| \begin{pmatrix} \overline{\kappa \partial_{p'}^2 H^1(0)} & 0 \\ 0 & 0 \end{pmatrix} v \right\|_{\rho, \sigma} \right| \\ &\geq \left| l \|v\|_{\rho, \sigma} - \left\| \begin{pmatrix} \overline{\kappa \partial_{p'}^2 H^1(0)} & 0 \\ 0 & 0 \end{pmatrix} v \right\|_{\rho, \sigma} \right| \geq \left| l \|v\|_{\rho, \sigma} - \left\| \begin{pmatrix} \overline{\kappa \partial_{p'}^2 H^1(0)} \end{pmatrix} \right\|_{\rho, \sigma} \|v\|_{\rho, \sigma} \right| \\ &\geq \left| l \|v\|_{\rho, \sigma} - \left\| \begin{pmatrix} \kappa \partial_{p'}^2 H^1(0, q') \end{pmatrix} \right\|_{\rho, \sigma} \|v\|_{\rho, \sigma} \right| \geq \left| l - \frac{\kappa E}{\sigma^2} \right| \|v\|_{\rho, \sigma}. \end{aligned}$$

where we use in the last line Cauchy's inequality and the fact $\|H^1(0, q')\|_{\rho, \sigma} \leq E$. □

C Lemmas for Canonical Transformations

In what follows we present several estimates relevant to canonical changes of variables defined by the Lie series method. For $\chi(p', q') \in \mathcal{A}_{\rho, \sigma}$ define

$$\chi_{\rho, \sigma}^* \equiv \max \left(\left\| \frac{\partial \chi}{\partial p'} \right\|_{\rho, \sigma}, \left\| \frac{\partial \chi}{\partial q'} \right\|_{\rho, \sigma} \right).$$

Lemma C.1

For any $\chi, f \in \mathcal{A}_{\rho, \sigma}$ and positive $\delta < \sigma < \rho$ we have

1. $\|\{\chi, f\}\|_{\rho-\delta, \sigma-\delta} \leq 2(n+1) \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right) \|f\|_{\rho, \sigma}.$
2. $\|\{\chi, \{\chi, f\}\}\|_{\rho-\delta, \sigma-\delta} \leq 8(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}.$

Proof :

$$\boxed{\|\{\chi, f\}\|_{\rho-\delta, \sigma-\delta} \leq 2(n+1) \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right) \|f\|_{\rho, \sigma}}$$

By definition

$$\{\chi, f\} = \sum_i \left[\frac{\partial \chi}{\partial p'_i} \frac{\partial f}{\partial q'_i} - \frac{\partial \chi}{\partial q'_i} \frac{\partial f}{\partial p'_i} \right].$$

It follows

$$|\{\chi, f\}| \leq \sum_i \left[\left| \frac{\partial \chi}{\partial p'_i} \right| \left| \frac{\partial f}{\partial q'_i} \right| + \left| \frac{\partial \chi}{\partial q'_i} \right| \left| \frac{\partial f}{\partial p'_i} \right| \right] \leq \chi_{\rho, \sigma}^* \sum_i \left[\left| \frac{\partial f}{\partial q'_i} \right| + \left| \frac{\partial f}{\partial p'_i} \right| \right] \leq \chi_{\rho, \sigma}^* \frac{2(n+1)}{\delta} \|f\|_{\rho, \sigma}$$

which implies

$$\|\{\chi, f\}\|_{\rho, \sigma} \leq 2(n+1) \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right) \|f\|_{\rho, \sigma}.$$

$$\boxed{\|\{\chi, \{\chi, f\}\}\|_{\rho-\delta, \sigma-\delta} \leq 8(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}}$$

We have by definition

$$\begin{aligned} \{\chi, \{\chi, f\}\} &= \sum_i \left[\frac{\partial \chi}{\partial p'_i} \frac{\partial}{\partial q'_i} \{\chi, f\} - \frac{\partial \chi}{\partial q'_i} \frac{\partial}{\partial p'_i} \{\chi, f\} \right] \\ &= \sum_i \left[\frac{\partial \chi}{\partial p'_i} \frac{\partial}{\partial q'_i} \sum_j \left[\frac{\partial \chi}{\partial p'_j} \frac{\partial f}{\partial q'_j} - \frac{\partial \chi}{\partial q'_j} \frac{\partial f}{\partial p'_j} \right] - \frac{\partial \chi}{\partial q'_i} \frac{\partial}{\partial p'_i} \sum_j \left[\frac{\partial \chi}{\partial p'_j} \frac{\partial f}{\partial q'_j} - \frac{\partial \chi}{\partial q'_j} \frac{\partial f}{\partial p'_j} \right] \right] \\ &= \sum_{i,j} \left[\frac{\partial \chi}{\partial p'_i} \left[\frac{\partial^2 \chi}{\partial q'_i \partial p'_j} \frac{\partial f}{\partial q'_j} + \frac{\partial \chi}{\partial p'_j} \frac{\partial^2 f}{\partial q'_i \partial q'_j} - \frac{\partial^2 \chi}{\partial q'_i \partial q'_j} \frac{\partial f}{\partial p'_j} - \frac{\partial \chi}{\partial q'_i} \frac{\partial^2 f}{\partial q'_i \partial p'_j} \right] \right. \\ &\quad \left. - \frac{\partial \chi}{\partial q'_i} \left[\frac{\partial^2 \chi}{\partial p'_i \partial p'_j} \frac{\partial f}{\partial q'_j} + \frac{\partial \chi}{\partial p'_j} \frac{\partial^2 f}{\partial p'_i \partial q'_j} - \frac{\partial^2 \chi}{\partial p'_i \partial q'_j} \frac{\partial f}{\partial p'_j} - \frac{\partial \chi}{\partial q'_i} \frac{\partial^2 f}{\partial p'_i \partial p'_j} \right] \right] \end{aligned}$$

There are eight expressions in the sum, each can be estimated using Cauchy's inequality. The first two expressions are estimated below and the rest can be estimated the same way.

$$\left\| \frac{\partial \chi}{\partial p'_i} \frac{\partial^2 \chi}{\partial q'_i \partial p'_j} \frac{\partial f}{\partial q'_j} \right\|_{\rho-\delta, \sigma-\delta} \leq \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma},$$

$$\left\| \frac{\partial \chi}{\partial p'_i} \frac{\partial \chi}{\partial p'_j} \frac{\partial^2 f}{\partial q'_i \partial q'_j} \right\|_{\rho-\delta, \sigma-\delta} \leq \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}.$$

Each of the eight expressions in the summation represents $(n+1)^2$ terms. Therefore the estimate gives

$$\|\{\chi, \{\chi, f\}\}\|_{\rho-\delta, \sigma-\delta} \leq 8(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}.$$

□

We now review the Lie series method in which the desired canonical transformation ϕ is given by the time one flow of a Hamiltonian χ called the generating function. If one evaluates a function f on the solutions of Hamilton's equations generated by χ one can write $\frac{df}{dt} = \{\chi, f\}$. That same functions f defined on the phase space is transformed $f \mapsto \mathcal{U}f = f \circ \phi$. Therefore this transformation of f can be written as $\mathcal{U}f = f + \{\chi, f\} + \frac{1}{2}\{\chi, \{\chi, f\}\} + \dots$ which is exactly the Taylor expansion of f . The transformation of a function can thus be expressed entirely in terms of the generating Hamiltonian χ without knowing the flow. The following lemma gives the conditions under which such a canonical transformation exists and several useful estimates. These estimates are key in the iterative lemma where the size of the perturbation after applying the canonical transformation must be known.

Lemma C.2

Suppose $\chi(p', q') \in \mathcal{A}_{\rho, \sigma}$ and its derivatives also belong to $\mathcal{A}_{\rho, \sigma}$ for given positive numbers ρ, σ . Consider the corresponding Hamiltonian system

$$\dot{p}' = -\frac{\partial \chi}{\partial q'}, \quad \dot{q}' = \frac{\partial \chi}{\partial p'}.$$

Assume that for some positive $\delta < \sigma$, $\delta < \rho$

$$\chi_{\rho, \sigma}^* \equiv \max \left(\left\| \frac{\partial \chi}{\partial p'} \right\|_{\rho, \sigma}, \left\| \frac{\partial \chi}{\partial q'} \right\|_{\rho, \sigma} \right) < \frac{\delta}{2}.$$

Then, for all initial conditions $(P', Q') \in D_{\rho-\delta, \sigma-\delta}$, the image of a point $(P', Q') \in D_{\rho-\delta, \sigma-\delta}$ is contained in $D_{\rho, \sigma}$, thus defining a canonical analytic transformation

$$\phi : D_{\rho-\delta, \sigma-\delta} \rightarrow D_{\rho, \sigma}, \quad \phi \in \mathcal{A}_{\rho-\delta, \sigma-\delta}.$$

The operator

$$\begin{aligned} \mathcal{U} : \mathcal{A}_{\rho, \sigma} &\rightarrow \mathcal{A}_{\rho-\delta, \sigma-\delta}, \\ f &\rightarrow \mathcal{U}f \equiv f \circ \phi, \end{aligned}$$

is then well defined and one has the estimates

1. $\|\phi - \text{identity}\|_{\rho-\delta, \sigma-\delta} \leq \chi_{\rho, \sigma}^*$,
2. $\|\mathcal{U}f\|_{\rho-\delta, \sigma-\delta} \leq \|f\|_{\rho, \sigma}$,
3. $\|\mathcal{U}f - f\|_{\rho-\delta, \sigma-\delta} \leq 4(n+1) \frac{\chi_{\rho, \sigma}^*}{\delta} \|f\|_{\rho, \sigma}$,
4. $\|\mathcal{U}f - f - \{\chi, f\}\|_{\rho-\delta, \sigma-\delta} \leq 32(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}$.

Proof :

One first shows the existence of solutions through an arbitrary initial condition $(P', Q') \in D_{\rho-\delta, \sigma-\delta}$. This follows if the vector field is Lipschitz throughout $D_{\rho-\delta, \sigma-\delta}$. Using Cauchy's inequality in the domain $D_{\rho-\delta/2, \sigma-\delta/2}$, the Lipschitz constant for the vector field

$$\dot{p}' = -\frac{\partial \chi}{\partial q'}, \quad \dot{q}' = \frac{\partial \chi}{\partial p'},$$

can be estimated as

$$\left\| \frac{\partial^2 \chi}{\partial q' \partial p'} \right\|_{\rho-\delta/2, \sigma-\delta/2} \leq 4(n+1) \frac{\chi_{\rho, \sigma}^*}{\delta}.$$

Next we argue that this local solution exists for $t = 1$, and does not leave $D_{\rho, \sigma}$ in that time. By assumption, the maximum velocity of the vector field is bounded by

$$\max(\|\dot{p}'\|, \|\dot{q}'\|) \leq \chi_{\rho, \sigma}^* < \frac{\delta}{2}.$$

Hence the maximum distance that a point in $D_{\rho-\delta, \sigma-\delta}$ can move in time $t = 1$ is $\frac{\delta}{2}$. Therefore

$$\phi^t : D_{\rho-\delta, \sigma-\delta} \rightarrow D_{\rho-\delta/2, \sigma-\delta/2} \subset D_{\rho, \sigma},$$

Next we prove the estimates.

1. $\|\phi - \text{identity}\|_{\rho-\delta, \sigma-\delta} \leq \chi_{\rho, \sigma}^*.$

This follows from the mean value theorem

$$\text{For } 0 \leq t \leq 1, \quad \|\phi - \text{identity}\|_{\rho-\delta, \sigma-\delta} = \|\phi^{t=1} - \phi^{t=0}\|_{\rho-\delta, \sigma-\delta} \leq \|\dot{\phi}^t\|(1-0) \leq \chi_{\rho, \sigma}^*.$$

2. $\|\mathcal{U}f\|_{\rho-\delta, \sigma-\delta} \leq \|f\|_{\rho, \sigma}.$

This is trivial, $\|\mathcal{U}f\|_{\rho-\delta, \sigma-\delta} \equiv \|f \circ \phi\|_{\rho-\delta, \sigma-\delta}$ but $\phi : D_{\rho-\delta, \sigma-\delta} \rightarrow D_{\rho, \sigma}$ which implies $\|f \circ \phi\|_{\rho-\delta, \sigma-\delta} \leq \|f\|_{\rho, \sigma}$.

3. $\|\mathcal{U}f - f\|_{\rho-\delta, \sigma-\delta} \leq 4(n+1) \frac{\chi_{\rho, \sigma}^*}{\delta} \|f\|_{\rho, \sigma}.$

By definition we have $\mathcal{U}f \equiv f \circ \phi^{t=1}$. Taylor expanding in t gives

$$f \circ \phi^t = f \circ \phi^0 + \frac{d}{dt}(f \circ \phi^t)|_{t=t'}, \quad 0 \leq t' \leq 1,$$

from which follows

$$\mathcal{U}f = f + \frac{df}{dt}|_{t=t'} \quad \text{or} \quad \mathcal{U}f - f = \frac{df}{dt}|_{t=t'} = \{\chi, f\}|_{t=t'}.$$

As argued above,

$$\phi(P', Q') \in D_{\rho-\delta/2, \sigma-\delta/2}, \quad 0 < \delta < 1, \quad (P', Q') \in D_{\rho-\delta, \sigma-\delta}.$$

Hence, using the estimate in Lemma C.1, with δ in the Lemma replaced by $\frac{\delta}{2}$, we obtain

$$\|\mathcal{U}f - f\|_{\rho-\delta, \sigma-\delta} \leq 4(n+1) \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right) \|f\|_{\rho, \sigma}.$$

$$4. \quad \|\mathcal{U}f - f - \{\chi, f\}\|_{\rho-\delta, \sigma-\delta} \leq 32(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}.$$

Taylor expanding in t

$$f \circ \phi^t = f \circ \phi^0 + \frac{df}{dt}t + \frac{1}{2} \frac{d^2 f}{dt^2} t^2 + \dots,$$

and applying the mean value theorem gives

$$\mathcal{U}f - f - \frac{df}{dt}|_{t=t''} = \frac{1}{2} \frac{d^2 f}{dt^2}|_{t=t''} = \frac{1}{2} \{\chi, \{\chi, f\}\}|_{t=t''}.$$

As above,

$$\phi(P', Q') \in D_{\rho-\delta/2, \sigma-\delta/2}, \quad 0 < \delta < 1, \quad (P', Q') \in D_{\rho-\delta, \sigma-\delta}.$$

Using this and Lemma C.1, with δ in the Lemma replaced by $\frac{\delta}{2}$, gives

$$\|\mathcal{U}f - f - \{\chi, f\}\|_{\rho-\delta, \sigma-\delta} \leq 32(n+1)^2 \left(\frac{\chi_{\rho, \sigma}^*}{\delta} \right)^2 \|f\|_{\rho, \sigma}.$$

□

D P.D.E on a Torus

Proof of Lemma 8.1

We write the coefficients of the Fourier series as follows

$$g_k = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q) e^{-ik \cdot q} dq_1 \cdots dq_n.$$

Since $G \in \mathcal{A}_\rho$, we can shift the path of integration as follows

$$\begin{aligned} g_k &= \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho) \left\{ \prod_{j=1}^n e^{-ik_j (q_j - i \frac{k_j}{|k_j|} \rho)} \right\} dq_1 \cdots dq_n \\ &= \frac{e^{-|k|\rho}}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho) \cdot e^{-ik \cdot q} dq \end{aligned}$$

(If $k_i \neq 0$ we don't shift in that direction) and $|g_k| \leq e^{-|k|\rho} \|G(q)\|_\rho$. □

Proof of Lemma 8.2

We begin by expanding F and G in Fourier Series

$$F(q) = \sum_{k \in \mathbb{Z}^n} f_k e^{ik \cdot q}, \quad G(q) = \sum_{k \in \mathbb{Z}^n \setminus 0} g_k e^{ik \cdot q},$$

substituting in (8.1) gives

$$\sum_i \lambda_i \sum_{k \in \mathbb{Z}^n} ik_i f_k e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n \setminus 0} g_k e^{ik \cdot q},$$

or

$$\sum_{k \in \mathbb{Z}^n} i\lambda \cdot k f_k e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n \setminus 0} g_k e^{ik \cdot q},$$

which implies

$$f_k = \frac{g_k}{i\lambda \cdot k}, \quad k \in \mathbb{Z}^n \setminus 0.$$

From this expression we see why we must require $\overline{G} = 0$ in order to construct a solution. Recall that $\lambda \in \Omega_\Gamma$ so that $\lambda \cdot k \neq 0$. We next prove convergence of

$$\sum_{k \in \mathbb{Z}^n \setminus 0} f_k e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{g_k}{i\lambda \cdot k} e^{ik \cdot q}.$$

From lemma 8.1 since $G \in \mathcal{A}_\rho$ it follows $|g_k| \leq \|G\|_\rho e^{-|k|\rho}$, and the following estimates, using $\|k\| \leq |k|$, hold

$$|\lambda \cdot k| \geq \Gamma \|k\|^{-n}, \quad |\lambda \cdot k| \geq \Gamma |k|^{-n}.$$

Using lemma A.3 gives

$$|k|^n \leq \left(\frac{n}{e\delta}\right)^n e^{|k|\delta}.$$

Hence, we use these estimates to estimate the Fourier coefficients as follows

$$|f_k| = \left| \frac{g_k}{\lambda \cdot k} \right| \leq \frac{\|G\|_\rho}{\Gamma} |k|^n e^{-|k|\rho} \leq \frac{\|G\|_\rho}{\Gamma} \left(\frac{n}{e\delta}\right)^n e^{-|k|(\rho-\delta)} = C e^{-|k|(\rho-\delta)}$$

where

$$C = \frac{\|G\|_\rho}{\Gamma} \left(\frac{n}{e\delta}\right)^n.$$

Using lemma A.2 replacing ρ with $\rho - \delta$ we obtain

$$F \in \mathcal{A}_{\rho-2\delta}, \quad \|F\|_{\rho-2\delta} \leq C \left(\frac{4}{\delta}\right)^n.$$

Next we get the first estimate on the norm of F in terms of the norm of G . The previous estimate can be rewritten as

$$\|F\|_{\rho-2\delta} \leq \Gamma^{-1} \left(\frac{16n}{e4\delta^2}\right)^n \|G\|_\rho = \left(\frac{16n}{e}\right)^n \frac{\|G\|_\rho}{\Gamma(2\delta)^{2n}}.$$

We have

$$\varpi = 2^{4n+1} \left(\frac{n+1}{e}\right)^{n+1} \geq \left(\frac{16n}{e}\right)^n,$$

since

$$\left(\frac{n+1}{e}\right)^{n+1} \geq \left(\frac{n}{e}\right)^n$$

Using this inequality, the estimate can be written as

$$\|F\|_{\rho-2\delta} \leq \frac{\varpi}{\Gamma(2\delta)^{2n}} \|G\|_\rho.$$

Letting $\delta \rightarrow \frac{\delta}{2}$ gives the result.

Finally, we prove the last estimate in the lemma. Recall that

$$F(q) = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{g_k}{i\lambda \cdot k} e^{ik \cdot q},$$

then, formally, we have

$$\frac{\partial F}{\partial q_j} = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{g_k}{i\lambda \cdot k} (ik_j) e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n \setminus 0} \frac{g_k}{\lambda \cdot k} (k_j) e^{ik \cdot q}.$$

We denote the Fourier coefficients of $\frac{\partial F}{\partial q_j}$ by $h_{k_j} = \frac{g_k}{\lambda \cdot k} (k_j)$, $j = 1, \dots, n$. Therefore $|h_{k_j}| \leq \Gamma^{-1} \|G\|_\rho |k|^{n+1} e^{-|k|\rho}$. From lemma A.3 we have

$$|h_{k_j}| \leq \Gamma^{-1} \|G\|_\rho \left(\frac{n+1}{e\delta} \right)^{n+1} e^{-|k|(\rho-\delta)}.$$

Form lemma A.2 we have

$$\left\| \frac{\partial F}{\partial q} \right\|_{\rho-2\delta} \leq C \left(\frac{4}{\delta} \right)^n,$$

where

$$C = \frac{1}{\Gamma \delta^{n+1}} \left(\frac{n+1}{e} \right)^{n+1} \|G\|_\rho,$$

so

$$\left\| \frac{\partial F}{\partial q} \right\|_{\rho-2\delta} \leq \frac{4^n}{\Gamma \delta^{n+1}} \left(\frac{n+1}{e} \right)^{n+1} \|G\|_\rho.$$

Let $\delta \rightarrow \frac{\delta}{2}$, then the estimate takes the form

$$\left\| \frac{\partial F}{\partial q} \right\|_{\rho-\delta} \leq \frac{4^n 2^{2n+1}}{\Gamma \delta^{n+1}} \left(\frac{n+1}{e} \right)^{n+1} \|G\|_\rho. \quad \square$$

E Fourier Transform Lemmas

We present some important results concerning a function and its Fourier transform defined on complex domains.

Lemma E.1

Suppose $f(z)$, $z = x + iy$, is analytic for $-a < y < b$ where $a > 0, b > 0$. In any strip in the interior of $-a < y < b$, let

$$f(z) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)x}) & (x \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)x}) & (x \rightarrow -\infty) \end{cases},$$

for every positive ε , where $\nu > 0, \mu > 0$. Then there exists a $\delta > \varepsilon$ such that $F(\omega)$, $\omega = u + iv$, defined by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-i\omega\zeta} d\zeta,$$

is analytic in the strip $-\mu + \delta \leq v \leq \nu - \delta$, satisfies

$$F(\omega) = \begin{cases} \mathcal{O}(e^{-(b-\varepsilon)u}) & (u \rightarrow \infty) \\ \mathcal{O}(e^{(a-\varepsilon)u}) & (u \rightarrow -\infty) \end{cases},$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{iz\omega} d\omega, \quad (\text{E.1})$$

for every z in the strip $-a < y < b$.

Proof: We have

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-i\zeta\omega} d\zeta,$$

to find the domain of analyticity of $F(\omega)$ we consider the sequence of functions $\{F_n(\omega)\}_{n=1}^{n=\infty}$ where

$$F_n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-n}^n f(\zeta) e^{-i\zeta\omega} d\zeta$$

and show that $\{F_n(\omega)\}_{n=1}^{n=\infty}$ converges to $F(\omega)$ uniformly. Consequently, since each of $F_n(\omega)$ is analytic in the strip $-\mu < v < \nu$, uniform convergence will imply $F(\omega)$ is analytic in some strip to be determined. Let $n = \max(c_1, |c_2|)$ where c_1 and c_2 are the constants in the definition of a function of exponential order in the real limit. Then, taking integrals along the real line,

$$\begin{aligned} |F_n(\omega) - F(\omega)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-n}^n f(\zeta) e^{-i\zeta\omega} d\zeta - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-i\zeta\omega} d\zeta \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_n^{\infty} f(\zeta) e^{-i\zeta\omega} d\zeta - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-n} f(\zeta) e^{-i\zeta\omega} d\zeta \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left[\int_n^{\infty} |f(\zeta)| |e^{-i\zeta\omega}| d\zeta + \int_{-\infty}^{-n} |f(\zeta)| |e^{-i\zeta\omega}| d\zeta \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_n^{\infty} |f(\zeta)| |e^{-i\zeta u}| |e^{\zeta v}| d\zeta + \int_{-\infty}^{-n} |f(\zeta)| |e^{-i\zeta u}| |e^{\zeta v}| d\zeta \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_n^{\infty} |f(\zeta)| e^{\zeta v} d\zeta + \int_{-\infty}^{-n} |f(\zeta)| e^{\zeta v} d\zeta \right] \\ &\leq \frac{1}{\sqrt{2\pi}} \left[C_1 \int_n^{\infty} e^{-(\nu-\varepsilon)\zeta} e^{\zeta v} d\zeta + C_2 \int_{-\infty}^{-n} e^{(\mu-\varepsilon)\zeta} e^{\zeta v} d\zeta \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{C_1 e^{-(\nu-\varepsilon-v)\zeta}}{-(\nu-\varepsilon-v)} \Big|_n^{\infty} + \frac{C_2 e^{(\mu-\varepsilon+v)\zeta}}{\mu-\varepsilon+v} \Big|_{-\infty}^{-n} \right]. \end{aligned}$$

The maximum finite bound for the expression above can be obtained by picking for the zeta intervals $[n, \infty)$ and $(-\infty, -n]$, $v = \nu - \delta$ and $v = -\mu + \delta$ respectively where $\delta > \varepsilon$. Note that, if v is picked anywhere between $(\nu - \delta, \nu]$ or $[-\mu, -\mu + \delta)$, the expression above becomes unbounded. On the other hand if v is chosen anywhere on $-\mu + \delta \leq v \leq \nu - \delta$ the expression above is bounded. Therefore

$$|F_n(\omega) - F(\omega)| \leq \frac{2C_3}{\sqrt{2\pi}} \frac{e^{-(\delta-\varepsilon)n}}{(\delta-\varepsilon)} < \epsilon$$

for $-\mu + \delta \leq v \leq \nu - \delta$ where $C_3 = \max(C_1, C_2)$. We can solve for n

$$-(\delta - \varepsilon)n < \ln \left[\frac{\sqrt{2\pi}}{2C_3} (\delta - \varepsilon) \epsilon \right]$$

so

$$n > \ln \left[\frac{\sqrt{2\pi}}{2C_3} (\delta - \varepsilon) \epsilon \right]^{-\frac{1}{(\delta-\varepsilon)}}.$$

We may take

$$N(\epsilon) = \ln \left[\frac{\sqrt{2\pi}}{2C_3} (\delta - \epsilon) \epsilon \right]^{-\frac{1}{(\delta - \epsilon)}}.$$

Therefore, for any $\epsilon > 0$ we have defined a $N(\epsilon)$, independent of ω , such that $|F(\omega) - F_n(\omega)| < \epsilon$, for all $n > N$. Consequently the series $\{F_n(\omega)\}$ converges uniformly to $F(\omega)$ for $-\mu + \delta \leq v \leq \nu - \delta$. Hence $F(\omega)$ is analytic in this strip. Next we prove the order results for $F(\omega)$. By applying Cauchy's theorem we may take the integral along any line of the strip parallel to the real axis. Thus

$$\begin{aligned} |F(\omega)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi - i\eta) e^{-i(\xi - i\eta)(u + iv)} d\xi \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(\xi - i\eta)| e^{-i\xi u} |e^{\xi v}| e^{-\eta u} |e^{-i\eta v}| d\xi \\ &= \frac{e^{-\eta u}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(\xi - i\eta)| e^{\xi v} d\xi \\ &= \frac{e^{-\eta u}}{\sqrt{2\pi}} \left[\int_0^{c_1} |f(\xi - i\eta)| e^{\xi v} d\xi + \int_{c_1}^{\infty} |f(\xi - i\eta)| e^{\xi v} d\xi \right. \\ &\quad \left. + \int_{c_2}^0 |f(\xi - i\eta)| e^{\xi v} d\xi + \int_{-\infty}^{c_2} |f(\xi - i\eta)| e^{\xi v} d\xi \right] \\ &\leq \frac{e^{-\eta u}}{\sqrt{2\pi}} \left[C_{c_1}(v) + C_1 \int_{c_1}^{\infty} e^{-(\nu - \epsilon)\xi} e^{\xi v} d\xi + C_{c_2}(v) + C_2 \int_{-\infty}^{c_2} e^{(\mu - \epsilon)\xi} e^{\xi v} d\xi \right] \\ &= \frac{e^{-\eta u}}{\sqrt{2\pi}} \left[C_{c_1}(v) + C_{c_2}(v) + \frac{C_1 e^{-(\nu - \epsilon - v)\xi}}{-(\nu - \epsilon - v)} \Big|_{c_1}^{\infty} + \frac{C_2 e^{(\mu - \epsilon + v)\xi}}{\mu - \epsilon + v} \Big|_{-\infty}^{c_2} \right], \end{aligned}$$

where

$$C_{c_1}(v) = \int_0^{c_1} |f(\xi - i\eta)| e^{\xi v} d\xi, \quad C_{c_2}(v) = \int_0^{c_2} |f(\xi - i\eta)| e^{\xi v} d\xi.$$

For the interval $[c_1, \infty)$ pick $v = \nu - \delta$ with $\epsilon < \delta$ such that

$$\frac{e^{-(\nu - \epsilon - v)\xi}}{-(\nu - \epsilon - v)} \Big|_{c_1}^{\infty} \leq \frac{e^{-(\delta - \epsilon)c_1}}{\delta - \epsilon}.$$

Similarly for the interval $(-\infty, c_2]$ pick $v = -\mu + \delta$ and with $\epsilon < \delta$ we have

$$\frac{e^{(\mu - \epsilon + v)\xi}}{\mu - \epsilon + v} \Big|_{-\infty}^{c_2} \leq \frac{e^{(\delta - \epsilon)c_2}}{\delta - \epsilon}.$$

We bound $C_{c_1}(v)$ and $C_{c_2}(v)$ by choosing $v = \nu - \delta$ and $v = -\mu + \delta$ respectively so that

$$\begin{aligned} C_{c_1}(v) &\leq \int_0^{c_1} |f(\xi - i\eta)| e^{(\nu - \delta)\xi} d\xi = C_{c_1}, \\ C_{c_2}(v) &\leq \int_{c_2}^0 |f(\xi - i\eta)| e^{(-\mu + \delta)\xi} d\xi = C_{c_2}. \end{aligned}$$

Consequently, with $c_3 = \min(c_1, |c_2|)$, $C_3 = \max(C_1, C_2)$ and $C_{c_3} = \max(C_{c_1}, C_{c_2})$

$$|F(\omega)| \leq \frac{2e^{-\eta u}}{\sqrt{2\pi}} \left[C_{c_3} + \frac{C_3 e^{-(\delta - \epsilon)c_3}}{(\delta - \epsilon)} \right],$$

and by taking η arbitrarily near to $-a$ and b the order results for $F(\omega)$ follow. (E.1) can be proved directly by the theorem of residues. Let $-a < \alpha < y < \beta < b$. Then

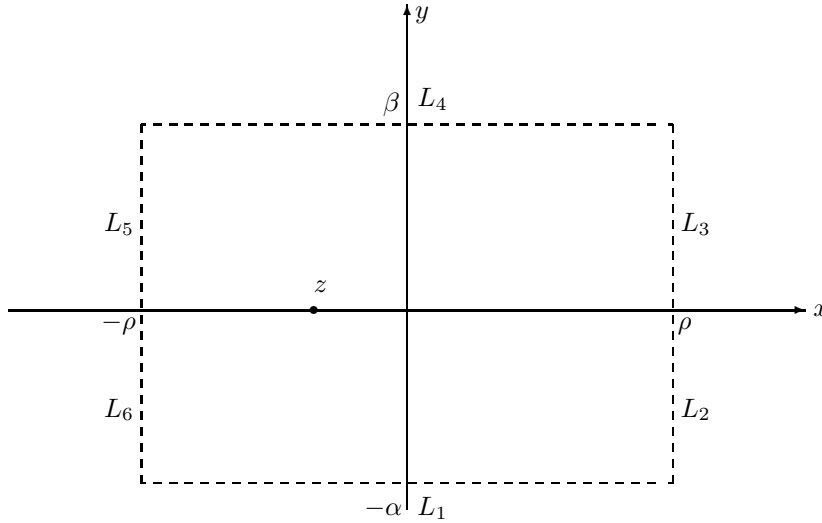
$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^\infty F(\omega) e^{iz\omega} d\omega &= \frac{1}{2\pi} \int_0^\infty e^{iz\omega} d\omega \int_{-i\alpha-\infty}^{-i\alpha+\infty} f(\zeta) e^{-i\zeta\omega} d\zeta = \frac{1}{2\pi} \int_{-i\alpha-\infty}^{-i\alpha+\infty} f(\zeta) d\zeta \int_0^\infty e^{-i(\zeta-z)\omega} d\omega \\ &= \frac{1}{2\pi i} \int_{-i\alpha-\infty}^{-i\alpha+\infty} \frac{f(\zeta)}{\zeta-z} d\zeta \end{aligned}$$

Similarly,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 F(\omega) e^{iz\omega} d\omega = \frac{1}{2\pi i} \int_{i\beta-\infty}^{i\beta+\infty} \frac{f(\zeta)}{z-\zeta} d\zeta.$$

Now we consider the following counter clockwise contour integral along a rectangle

$$\int_C \frac{f(\zeta)}{\zeta-z} d\zeta = \int_{i\beta+\rho}^{i\beta-\rho} \frac{f(\zeta)}{\zeta-z} d\zeta + \int_{L_5} \frac{f(\zeta)}{\zeta-z} d\zeta + \int_{L_6} \frac{f(\zeta)}{\zeta-z} d\zeta + \int_{i\alpha-\rho}^{i\alpha+\rho} \frac{f(\zeta)}{\zeta-z} d\zeta + \int_{L_2} \frac{f(\zeta)}{\zeta-z} d\zeta + \int_{L_3} \frac{f(\zeta)}{\zeta-z} d\zeta.$$



We show the contribution from each of L_2, L_3, L_5, L_6 is zero. For L_2 we use the following parameterization $\zeta(y) = \rho + iy$; $-\alpha \leq y \leq 0$ and since $f(\zeta)$ must be bounded for all ζ by some real number A

$$\left| \frac{f(\zeta)}{\zeta-z} \right| \leq \frac{A}{|\rho + iy - z|}$$

so

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_2} \frac{f(\zeta)}{\zeta-z} d\zeta \right| \leq \lim_{\rho \rightarrow \infty} \frac{A\alpha}{|\rho + iy - z|} = 0.$$

A similar procedure with $0 \leq y \leq \beta$ in the parameterization gives

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_3} \frac{f(\zeta)}{\zeta-z} d\zeta \right| \leq \lim_{\rho \rightarrow \infty} \frac{A\alpha}{|\rho + iy - z|} = 0.$$

For L_5 we use the following parameterization $\zeta(y) = -\rho + iy$, $0 \leq y \leq \beta$. and

$$\left| \frac{f(\zeta)}{\zeta - z} \right| \leq \frac{A}{|-\rho + iy - z|}$$

so

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_5} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \lim_{\rho \rightarrow \infty} \frac{A\alpha}{|-\rho + iy - z|} = 0.$$

A similar procedure with $-\alpha \leq y \leq 0$ in the parameterization gives

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_6} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \lim_{\rho \rightarrow \infty} \frac{A\alpha}{|-\rho + iy - z|} = 0.$$

Therefore

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{i\beta+\rho}^{i\beta-\rho} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{-i\alpha-\rho}^{-i\alpha+\rho} \frac{f(\zeta)}{\zeta - z} d\zeta$$

and

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-iz\omega} d\omega = \frac{1}{2\pi i} \left[\int_{i\beta-\infty}^{i\beta+\infty} \frac{f(\zeta)}{z - \zeta} d\zeta + \int_{-i\alpha-\infty}^{-i\alpha+\infty} \frac{f(\zeta)}{\zeta - z} d\zeta \right] \\ &= \frac{1}{2\pi i} \left[\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \right] = \frac{1}{2\pi i} [2\pi i f(z)] = f(z). \end{aligned}$$

□

Lemma E.2

Suppose $F(\omega)$, $\omega = u + iv$, is analytic, regular for $-\nu < v < \mu$ where $\nu > 0, \mu > 0$. In any strip in the interior to $-\nu < v < \mu$, let

$$F(\omega) = \begin{cases} \mathcal{O}(e^{-(b-\varepsilon)u}) & (u \rightarrow \infty) \\ \mathcal{O}(e^{(a-\varepsilon)u}) & (u \rightarrow -\infty) \end{cases},$$

for every positive ε , where $a > 0, b > 0$. Then there exists a $\delta > 0$ such that $f(z)$, $z = x + iy$, defined by

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\zeta) e^{iz\zeta} d\zeta,$$

is analytic for $-b + \delta \leq y \leq a - \delta$, satisfies

$$f(z) = \begin{cases} \mathcal{O}(e^{-(\mu-\varepsilon)x}) & (x \rightarrow \infty) \\ \mathcal{O}(e^{(\nu-\varepsilon)x}) & (x \rightarrow -\infty) \end{cases},$$

and

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{-i\omega z} dz,$$

for every ω in the strip $-\nu < v < \mu$.

Proof: We have

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\zeta) e^{iz\zeta} d\zeta,$$

and define the sequence $\{f_n(z)\}$ where

$$f_n(z) = \frac{1}{\sqrt{2\pi}} \int_{-n}^n F(\zeta) e^{iz\zeta} d\zeta.$$

Similar to Theorem E.1 we can show that this sequence converges uniformly to $f(z)$ for $-b + \delta \leq y \leq a - \delta$. And since each of $f_n(z)$ is analytic for $-b + \delta \leq y \leq a - \delta$, it follows that $f(z)$ is analytic in this strip. Next we prove the order results on $f(z)$. Applying Cauchy's theorem we may take the integral along any line of the strip parallel to the real axis. Thus

$$\begin{aligned} |f(z)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi + i\eta) e^{i(x+iy)(\xi+i\eta)} d\xi \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |F(\xi + i\eta)| |e^{ix\xi}| |e^{-x\eta}| |e^{-y\xi}| |e^{-iy\eta}| d\xi \\ &\leq \frac{e^{-x\eta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |F(\xi + i\eta)| e^{-y\xi} d\xi \\ &= \frac{e^{-x\eta}}{\sqrt{2\pi}} \left[\int_0^{c_1} |F(\xi + i\eta)| e^{-y\xi} d\xi + \int_{c_1}^{\infty} |F(\xi + i\eta)| e^{-y\xi} d\xi \right. \\ &\quad \left. + \int_{c_2}^0 |F(\xi + i\eta)| e^{-y\xi} d\xi + \int_{-\infty}^{c_2} |F(\xi + i\eta)| e^{-y\xi} d\xi \right] \\ &\leq \frac{e^{-x\eta}}{\sqrt{2\pi}} \left[C_{c_1}(y) + C_1 \int_{c_1}^{\infty} e^{-(b-\varepsilon)\xi} e^{-y\xi} d\xi + C_{c_2}(y) + C_2 \int_{-\infty}^{c_2} e^{(a-\varepsilon)\xi} e^{-y\xi} d\xi \right] \\ &= \frac{e^{-x\eta}}{\sqrt{2\pi}} \left[C_{c_1}(y) + C_{c_2}(y) + \frac{C_1 e^{-(b-\varepsilon+y)\xi}}{-(b-\varepsilon+y)} \Big|_{c_1}^{\infty} + \frac{C_2 e^{(a-\varepsilon-y)\xi}}{(a-\varepsilon-y)} \Big|_{-\infty}^{c_2} \right]. \end{aligned}$$

To bound $C_{c_1}(y)$ and $C_{c_2}(y)$ we chose $y = -b + \delta$ and $y = a - \delta$ respectively so that

$$\begin{aligned} C_{c_1}(y) &\leq \int_0^{c_1} |F(\xi + i\eta)| e^{(b-\delta)\xi} d\xi = C_{c_1}, \\ C_{c_2}(y) &\leq \int_{c_2}^0 |F(\xi + i\eta)| e^{-(a-\delta)\xi} d\xi = C_{c_2}. \end{aligned}$$

For the ξ intervals $[c_1, \infty)$ and $(-\infty, c_2]$ we pick $y = -b + \delta$ and $y = a - \delta$ respectively with $\delta > \varepsilon$ and obtain with $c_3 = \min(c_1, |c_2|)$, $C_3 = \max(C_1, C_2)$, and $C_{c_3} = \max(C_{c_1}, C_{c_2})$

$$|f(z)| \leq \frac{2e^{-x\eta}}{\sqrt{2\pi}} \left[C_{c_3} + \frac{C_3 e^{-(\delta-\varepsilon)c_3}}{(\delta-\varepsilon)} \right]$$

and by choosing η arbitrarily close to $-\nu$ and μ the order results follow. The rest of the theorem can be proven with the theory of residues exactly like the previous theorem except that the contour integral will be taken clockwise. \square

Lemma E.3

Given $G(q, t) \in \mathcal{A}_\rho$ with

$$G(q, t) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q},$$

and if

$$g_k(t) = \begin{cases} \mathcal{O}(e^{-(b-\varepsilon)t_R}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(a-\varepsilon)t_R}) & (t_R \rightarrow -\infty) \end{cases},$$

then

$$g_k(t) = \begin{cases} \mathcal{O}(e^{-(b-\varepsilon)t_R}) \mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(a-\varepsilon)t_R}) \mathcal{O}(e^{-|k|\rho}) & (t_R \rightarrow -\infty) \end{cases}.$$

Proof:

$$g_k(t) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q, t) e^{-ik \cdot q} dq$$

where the integral is along real axis. Lift the integral as follows

$$g_k(t) = \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho) \cdot \left\{ \prod_{j=1}^n e^{-ik_j (q_j - i \frac{k_j}{|k_j|} \rho)} \right\} dq$$

$$g_k(t) = \frac{e^{-|k|\rho}}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho) \cdot e^{-ik \cdot q} dq$$

and $|g_k(t)| \leq e^{-|k|\rho} \|G(q, t)\|_\rho$. We know

$$\|G(q, t)\|_\rho = \sup_{q \in \mathcal{D}_\rho} |G(q, t)|.$$

Let q^* be arbitrary in the interior of the strip $|q_I| \leq \rho$. Then by the above and the assumption on the order of $g_k(t)$

$$\begin{aligned} |G(q^*, t)| &= \left| \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q^*} \right| \leq \sum_{k \in \mathbb{Z}^n} C e^{-At_R} e^{-|k|\rho} |e^{ik \cdot q^*}| \leq C e^{-At_R} \sum_{k \in \mathbb{Z}^n} e^{-|k|\rho} e^{-k \cdot q_I^*} \leq C e^{-At_R} \sum_{k \in \mathbb{Z}^n} e^{-|k|(\rho - q_I^*)} \\ &\leq C e^{-At_R} \sum_{k \in \mathbb{Z}^n} e^{-|k|(\rho - (\rho - \epsilon))} \leq C e^{-At_R} \sum_{k \in \mathbb{Z}^n} e^{-|k|\epsilon} \leq C e^{-At_R} 2^n \sum_{\substack{k \in \mathbb{Z}^n \\ k_1, \dots, k_n \geq 0}} e^{-\epsilon \sum k_j} \\ &= C e^{-At_R} 2^n \left(\sum_{k=0}^{\infty} e^{-\epsilon k} \right)^n = C e^{-At_R} 2^n \left(\frac{1}{1 - e^{-\epsilon}} \right)^n, \end{aligned}$$

where A is $b - \varepsilon$ or $-a + \varepsilon$. \square

The following lemma shows that under certain conditions we are assured the existence of the complex Fourier transform.

Lemma E.4

Suppose $f(t)$, $t = t_R + it_I$, is analytic for $-a < t_I < b$ where $a > 0, b > 0$. In any strip in the interior of $-a < t_I < b$, let

$$f(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R}) & (t_R \rightarrow \infty) \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R}) & (t_R \rightarrow -\infty) \end{cases},$$

for every positive ε , where $\nu > 0, \mu > 0$. Assume the Fourier transform exists

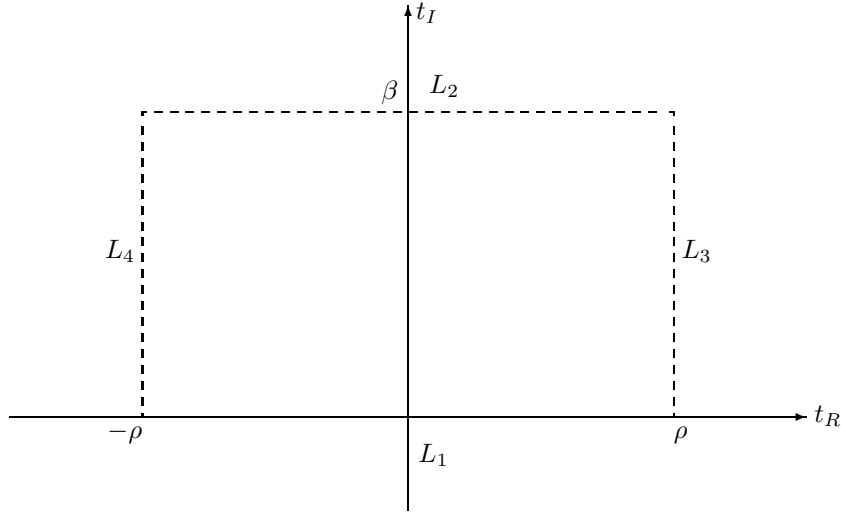
$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-i\omega\zeta} d\zeta < \infty.$$

Then the complex Fourier transform defined as

$$F^c(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\beta}^{\infty+i\beta} f(\zeta) e^{-i\omega\zeta} d\zeta,$$

with $-a < \beta < b$ exists and $F^c(\omega) = F(\omega)$.

Proof: We consider the counter clockwise contour integral on the rectangle R show below



Since $f(t)$ is analytic for $-a < t_I < b$ we know the contour integral is zero. Consequently

$$-\int_{L_2} = \int_{L_1} + \int_{L_3} + \int_{L_4}.$$

We analyze the integral along L_3 using the parameterization $t = \rho + it_I$ such that

$$\int_{L_3} f_k(t) e^{-i\omega t} dt = \int_0^\beta f_k(\rho + it_I) e^{-i\omega(\rho + it_I)} i dt_I = \int_0^\beta f_k(\rho + it_I) e^{-i\omega\rho} e^{v\rho} e^{ut_I} e^{ivt_I} i dt_I$$

and

$$\left| \int_{L_3} f_k(t) e^{-i\omega t} dt \right| \leq \int_0^\beta |f_k(\rho + it_I)| e^{v\rho} e^{ut_I} dt_I.$$

Note although

$$|f_k(t)| \leq C_1 e^{-(\nu-\varepsilon)t_R} \quad c_1 \leq t_R < \infty,$$

we can choose a constant C_3 large enough such that

$$|f_k(t)| \leq C_3 e^{-(\nu-\varepsilon)t_R} \quad 0 \leq t_R < \infty,$$

and

$$\left| \int_{L_3} f_k(t) e^{-i\omega t} dt \right| \leq \int_0^\beta C_3 e^{-(\nu-\varepsilon)\rho} e^{v\rho} e^{ut_I} dt_I.$$

By Theorem E.1, $F(\omega)$ is analytic inside the strip $-\mu + \delta \leq v \leq \nu - \delta$ where $\varepsilon < \delta$ so that

$$\left| \int_{L_3} f_k(t) e^{-i\omega t} dt \right| \leq C_3 e^{-(\nu-\varepsilon)\rho} e^{(\nu-\delta)\rho} e^{|u|\beta} \beta.$$

Clearly

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_3} f_k(t) e^{-i\omega t} dt \right| = 0.$$

Similarly for L_4 using the parameterization $t = -\rho + it_I$

$$\int_{L_e} f_k(t) e^{-i\omega t} dt = \int_\beta^0 f_k(-\rho + it_I) e^{-i\omega(-\rho + it_I)} i dt_I = \int_\beta^0 f_k(-\rho + it_I) e^{iu\rho} e^{-v\rho} e^{ut_I} e^{ivt_I} i dt_I$$

and

$$\left| \int_{L_4} f_k(t) e^{-i\omega t} dt \right| \leq \int_\beta^0 |f_k(-\rho + it_I)| e^{-v\rho} e^{ut_I} dt_I.$$

With a large enough constant C_4

$$\left| \int_{L_4} f_k(t) e^{-i\omega t} dt \right| \leq \int_\beta^0 C_4 e^{-(\mu-\varepsilon)\rho} e^{-v\rho} e^{ut_I} dt_I \leq C_4 e^{-(\mu-\varepsilon)\rho} e^{(\mu-\delta)\rho} e^{|u|\beta} \beta$$

and clearly

$$\lim_{\rho \rightarrow \infty} \left| \int_{L_4} f_k(t) e^{-i\omega t} dt \right| = 0.$$

It follows

$$\lim_{\rho \rightarrow \infty} - \int_{L_2} = \lim_{\rho \rightarrow \infty} \int_{L_1}$$

or

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-i\omega\zeta} d\zeta = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\beta}^{\infty+i\beta} f(\zeta) e^{-i\omega\zeta} d\zeta = F^c(\omega). \quad \square$$

F Properties of p', q' -Exponential Form functions

In this section we study some characteristics of functions of the form

$$F(p', q') = f(p') + r(p', q') + g(p', q'),$$

$$r(p', q') = \sum_{k \in \mathbb{Z}^n} s_k(p') e^{ik \cdot q},$$

$$g(p', q') = \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

where $F(p', q') \in \mathcal{A}_{\rho, \sigma}$ and $e_k(t)$ is of exponential order with respect to time. We will refer to these functions as functions of p', q' -exponential form. The Hamiltonian under consideration will be of p', q' -exponential form. There are several things we want to know about these functions to carry out the iterative process of the KAM proof. First, we want to be able to extract from $F(p', q')$ the quasiperiodic and exponential-order-with-respect-to-time parts as functions of $F(p', q')$ itself. The importance of this will become evident in the next section. Also, we want to show the derivative with respect to p'_j or q'_j , $j = 1, \dots, n+1$, of a p', q' -exponential-form function is another p', q' -exponential-form function. First we prove an important characteristic of $g(p', q')$, the function which is of exponential order with respect to time.

Lemma F.1

Given a function $g(p', q') \in \mathcal{A}_{\rho, \sigma}$ of the form

$$g(p', q') = \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

where $e_k(t)$ is of exponential order with respect to time, $\overline{g}(0) = 0$.

Proof: We simply show the time average of $g(p', q')$ is zero

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T g(p', q') dt = \sum_{k \in \mathbb{Z}^n} h_k(p') \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T e_k(t) dt e^{ik \cdot q} = 0. \quad \square$$

Next we show how to express $g(0, q')$ in terms of $F(p', q')$.

Lemma F.2

Given a function $F(p', q') \in \mathcal{A}_{\rho, \sigma}$ of p', q' -exponential form as defined previously, $T(0, q') = F(0, q') - \overline{\overline{F}}(0)$ represents the q -quasiperiodic and exponential-order-with-respect-to-time parts of $F(0, q')$. Moreover $T(0, q')$ has zero average.

Proof: First

$$\overline{\overline{F}}(0) = \overline{f}(0) + \overline{r}(0) + \overline{g}(0) = f(0) + s_0(0).$$

Therefore

$$F(0, q') - \overline{\overline{F}}(0) = f(0) + r(0, q') + g(0, q') - f(0) - s_0(0) = \sum_{k \in \mathbb{Z}^n \setminus 0} s_k(0) e^{ik \cdot q} + g(0, q'). \quad \square$$

Next we will show the derivative with respect to p'_j or q'_j , $j = 1, \dots, n+1$, of a $p' - q'$ -exponential-form function is another $p' - q'$ -exponential-form function. It is easy to see this is true for the derivative with respect to p'_j $j = 1, \dots, n+1$.

Lemma F.3

Let $F(p', q')$ be defined as before. Then $\frac{\partial F}{\partial p'_j}(p', q')$ $j = 1, \dots, n+1$ are of p', q' -exponential-form and $\frac{\partial F}{\partial p'_j}(p', q') \in \mathcal{A}_{\rho, \sigma}$.

Proof: First we write out the derivative

$$\frac{\partial F(p', q')}{\partial p'_j} = \frac{\partial f(p')}{\partial p'_j} + \frac{\partial r(p', q')}{\partial p'_j} + \frac{\partial g(p', q')}{\partial p'_j} = \frac{\partial f(p')}{\partial p'_j} + \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k(p')}{\partial p'_j} e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k(p')}{\partial p'_j} e_k(t) e^{ik \cdot q}.$$

Furthermore we know, as consequence of Cauchy's Integral formula, that if a function is analytic in some domain then all its derivatives are analytic in the same domain. \square

The next lemma show the derivative of a $p' - q'$ -exponential-form function with respect to q'_j , $j = 1, \dots, n$, is another $p' - q'$ -exponential-form function.

Lemma F.4

Let $F(p', q')$ be defined as before. Then $\frac{\partial F}{\partial q'_j}(p', q')$, $j = 1, \dots, n$, are of p', q' -exponential-form and $\frac{\partial F}{\partial q'_j}(p', q') \in \mathcal{A}_{\rho, \sigma}$.

Proof: First we write out the derivative for $j = 1, \dots, n$

$$\frac{\partial F(p', q')}{\partial q'_j} = \frac{\partial r(p', q')}{\partial q'_j} + \frac{\partial g(p', q')}{\partial q'_j} = \sum_{k \in \mathbb{Z}^n} ik_j s_k(p') e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} ik_j h_k(p') e_k(t) e^{ik \cdot q}$$

Furthermore we know, as consequence of Cauchy's Integral formula, that if a function is analytic in some domain then all its derivatives are analytic in the same domain. \square

Recall a function $e_k(t)$ is said to be of exponential order with respect to time if the following holds

$$e_k(t) = \begin{cases} \mathcal{O}(e^{-(\nu-\varepsilon)t_R}) & ; \quad 0 \leq c_1 < t_R < \infty \\ \mathcal{O}(e^{(\mu-\varepsilon)t_R}) & ; \quad -\infty < t_R < c_2 < 0. \end{cases}$$

The following lemma examines the derivative of a $p' - q'$ -exponential-form function with respect to $q'_{n+1} = t$.

Lemma F.5

Let $F(p', q')$ be defined as before. Then for some positive $\delta > \sigma$, $\frac{\partial F}{\partial t}(p', q') \in \mathcal{A}_{\rho, \sigma-\delta}$ is of p', q' -exponential-form. Moreover for the n th derivative we have the following estimates

$$|e_k^{(n)}(t)| \leq \frac{C_1 n!}{\delta^n} e^{(\nu-\varepsilon)\delta} e^{-(\nu-\varepsilon)t_R}, \quad 0 < c_1 < t_R < \infty,$$

$$|e_k^{(n)}(t)| \leq \frac{C_2 n!}{\delta^n} e^{-(\mu-\varepsilon)\delta} e^{(\mu-\varepsilon)t_R}, \quad -\infty < t_R < c_2 < 0,$$

for all $t \in \mathcal{D}_{\rho, \sigma-\delta}$.

Proof: We will prove this using Cauchy's Integral formula. Since $e_k(t)$ is exponentially small with respect to time we have

$$|e_k(t)| \leq C_1 e^{-(\nu-\varepsilon)t_R}, \quad 0 < c_1 < t_R < \infty,$$

$$|e_k(t)| \leq C_2 e^{(\mu-\varepsilon)t_R}, \quad -\infty < t_R < c_2 < 0.$$

Since $e_k(t)$ is analytic in the interior of the strip $-\sigma \leq t_{Im} \leq \sigma$ we have

$$e_k^{(n)}(t) = \frac{n!}{2\pi i} \int_C \frac{e_k(\omega)}{(\omega - t)^{n+1}} d\omega$$

where C is the circle of radius δ centered at t . We used the parameterization $\omega = t - \delta e^{i\theta}$ $0 \leq \theta \leq 2\pi$ and obtain

$$e_k^{(n)}(t) = \frac{n!}{2\pi i} \int_C \frac{e_k(\omega)}{(\omega - t)^{n+1}} d\omega = \int_0^{2\pi} \frac{e_k(t_R - \delta \cos\theta + i(t_I - \delta \sin\theta))}{(\delta e^{i\theta})^{n+1}} i \delta e^{i\theta} d\theta$$

and for $0 < c_1 < t_R < \infty$

$$|e_k^{(n)}(t)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|e_k(t_R - \delta \cos \theta + i(t_I - \delta \sin \theta))|}{(\delta)^{n+1}} \delta d\theta \leq \frac{n!}{2\pi \delta^n} \int_0^{2\pi} C_1 e^{-(\nu-\varepsilon)(t_R - \delta \cos \theta)} d\theta \leq \frac{C_1 n!}{\delta^n} e^{(\nu-\varepsilon)\delta} e^{-(\nu-\varepsilon)t_R}.$$

Therefore

$$|e_k^{(n)}(t)| \leq \frac{C_1 n!}{\delta^n} e^{(\nu-\varepsilon)\delta} e^{-(\nu-\varepsilon)t_R}, \quad 0 < c_1 < t_R < \infty,$$

for all $t \in \mathcal{D}_{\rho, \sigma-\delta}$. Similarly for $-\infty < t_R < c_2 < 0$ with the parameterization $\omega = t + \delta e^{i\theta}$ we obtain

$$|e_k^{(n)}(t)| \leq \frac{C_2 n!}{\delta^n} e^{-(\mu-\varepsilon)\delta} e^{(\mu-\varepsilon)t_R}.$$

□

We want to show the product of two quasiperiodic functions is quasiperiodic.

Lemma F.6

Let $F(q), G(q) \in \mathcal{A}_\rho$ be periodic with respect to $q \in \mathbb{R}$. It follows $F(q)G(q) = H(q)$ is periodic with respect to q and $H(q)$ can be expressed as

$$\sum_{k \in \mathbb{Z}} c_k e^{ikq},$$

where c_k is defined in terms of the series expressions for F and G .

Proof: First we prove that the series expression for F and G converge absolutely to the functions for $q \in \mathcal{D}_\rho$. Given

$$F(q) \sim \sum_{k \in \mathbb{Z}} a_k e^{ikq} \in \mathcal{A}_\rho, \quad G(q) \sim \sum_{k \in \mathbb{Z}} b_k e^{ikq} \in \mathcal{A}_\rho$$

we have

$$|a_k| \leq \|F\|_\rho e^{-|k|\rho}.$$

Furthermore, given $|q_I| \leq \rho - \delta$ for some positive δ

$$|e^{ik(q_R + iq_I)}| \leq e^{-kq_I} \leq e^{|k|(\rho-\delta)}.$$

Therefore

$$|a_k e^{ikq}| \leq \|F\|_\rho e^{-|k|\rho} e^{|k|(\rho-\delta)} \leq \|F\|_\rho e^{-|k|\delta}.$$

A similar estimate holds for $|b_k e^{ikq}|$. Consequently, $\sum_{k \in \mathbb{Z}} a_k e^{ikq}$ and $\sum_{k \in \mathbb{Z}} b_k e^{ikq}$ converge absolutely to $F(q)$ and $G(q)$ respectively for $q \in \mathcal{D}_\rho$. Next we look at the product $H(q)$ given by

$$\begin{aligned} H(q) &= \left(\sum_{k \in \mathbb{Z}} a_k e^{ikq} \right) \left(\sum_{k \in \mathbb{Z}} b_k e^{ikq} \right) = \left(\sum_{k=-\infty}^{k=\infty} a_k e^{ikq} \right) \left(\sum_{k=-\infty}^{k=\infty} b_k e^{ikq} \right) = \sum_{n=-\infty}^{n=\infty} \left(\sum_{k=-\infty}^{k=\infty} a_k e^{ikq} b_n e^{iq(n-k)} \right) \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} a_k b_n e^{ikq} \right) e^{inq} = \sum_{n \in \mathbb{Z}} c_n e^{inq} \in \mathcal{A}_\rho. \quad \square \end{aligned}$$

A similar result holds for quasiperiodic functions and can be proven similarly as above. Given

$$F(q) = \sum_{k \in \mathbb{Z}^n} a_k e^{ik \cdot q}, \quad G(q) = \sum_{k \in \mathbb{Z}^n} b_k e^{ik \cdot q}.$$

It follows

$$H(q) = F(q)G(q) = \sum_{k \in \mathbb{Z}^n} \left(\sum_{n \in \mathbb{Z}^n} a_n b_k e^{in \cdot q} \right) e^{ik \cdot q} = \sum_{k \in \mathbb{Z}^n} c_k e^{ik \cdot q}.$$

Lemma F.7

Let $F(p', q') \in \mathcal{A}_{\rho, \sigma}$ be of the form

$$F(p', q') = f(p') + r(p', q') + g(p', q'),$$

$$r(p', q') = \sum_{k \in \mathbb{Z}^n} s_k(p') e^{ik \cdot q},$$

$$g(p', q') = \sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q},$$

where $e_k(t)$ is of exponential order with respect to time. Let $\chi(p', q') \in \mathcal{A}_{\rho', \sigma'}$, $\rho' < \rho$ and $\sigma' < \sigma$, be of the form

$$\chi(p', q') = X(q') + \xi \cdot q' + Y(q') \cdot p',$$

where

$$\begin{aligned} X(q, t) &= \mathcal{Y}(q) + \mathcal{T}(q'), \quad \mathcal{Y}(q) = \sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q}, \quad \mathcal{T}(q') = \sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \\ Y_j(q') &= \mathcal{S}_j(q) + \mathcal{F}_j(q'), \quad \mathcal{S}_j(q) = \sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q}, \quad \mathcal{F}_j(q') = \sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q}, \end{aligned}$$

and where ξ is a constant vector. It follows, given some positive $\delta < \sigma'$, $\{\chi, F\}(p', q') \in \mathcal{A}_{\rho' - \delta, \sigma' - \delta}$ is of p', q' -exponential form.

Proof:

We write out the expression

$$\begin{aligned} \{\chi, F\} &= \sum_{j=1}^{n+1} \left[\frac{\partial \chi}{\partial p'_j} \frac{\partial F}{\partial q'_j} - \frac{\partial \chi}{\partial q'_j} \frac{\partial F}{\partial p'_j} \right] \\ &= \sum_{j=1}^{n+1} \left[(Y_j(q')) \left(\frac{\partial r}{\partial q'_j}(p', q') + \frac{\partial g}{\partial q'_j}(p', q') \right) - \left(\frac{\partial X}{\partial q'_j}(q') + \xi_j + \sum_{l=1}^{n+1} \frac{\partial Y_l}{\partial q'_j}(q') p'_l \right) \right. \\ &\quad \cdot \left. \left(\frac{\partial f}{\partial p'_j}(p') + \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right) \right] \\ &= \sum_{j=1}^{n+1} \left[(\mathcal{S}_j(q) + \mathcal{F}_j(q')) \left(\frac{\partial r}{\partial q'_j}(p', q') + \frac{\partial g}{\partial q'_j}(p', q') \right) - \left(\frac{\partial}{\partial q'_j} (\mathcal{Y}(q) + \mathcal{T}(q')) + \xi_j + \sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} (\mathcal{S}_l(q) + \mathcal{F}_l(q')) p'_l \right) \right] \end{aligned}$$

$$\begin{aligned}
& \cdot \left(\frac{\partial f}{\partial p'_j}(p') + \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right) \Bigg] \\
&= \sum_{j=1}^{n+1} \left[\mathcal{S}_j(q) \frac{\partial r}{\partial q'_j}(p', q') + \mathcal{S}_j(q) \frac{\partial g}{\partial q'_j}(p', q') + \mathcal{F}_j(q') \frac{\partial r}{\partial q'_j}(p', q') + \mathcal{F}_j(q') \frac{\partial g}{\partial q'_j}(p', q') \right. \\
&- \left(\frac{\partial \mathcal{Y}}{\partial q'_j}(q) + \frac{\partial \mathcal{T}}{\partial q'_j}(q') + \xi_j + \sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j}(q) p'_l + \sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \right) \\
&\cdot \left(\frac{\partial f}{\partial p'_j}(p') + \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right) \Bigg] \\
&= \sum_{j=1}^{n+1} \left[\mathcal{S}_j(q) \frac{\partial r}{\partial q'_j}(p', q') + \mathcal{S}_j(q) \frac{\partial g}{\partial q'_j}(p', q') + \mathcal{F}_j(q') \frac{\partial r}{\partial q'_j}(p', q') + \mathcal{F}_j(q') \frac{\partial g}{\partial q'_j}(p', q') \right. \\
&- \left(\frac{\partial \mathcal{Y}}{\partial q'_j}(q) \frac{\partial f}{\partial p'_j}(p') + \frac{\partial \mathcal{Y}}{\partial q'_j}(q) \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \frac{\partial \mathcal{Y}}{\partial q'_j}(q) \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right. \\
&+ \frac{\partial \mathcal{T}}{\partial q'_j}(q') \frac{\partial f}{\partial p'_j}(p') + \frac{\partial \mathcal{T}}{\partial q'_j}(q') \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \frac{\partial \mathcal{T}}{\partial q'_j}(q') \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \\
&+ \xi_j \frac{\partial f}{\partial p'_j}(p') + \xi_j \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \xi_j \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \\
&+ \sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j}(q) p'_l \frac{\partial f}{\partial p'_j}(p') + \sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j}(q) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j}(q) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \\
&\left. + \sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \frac{\partial f}{\partial p'_j}(p') + \sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} + \sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right) \Bigg].
\end{aligned}$$

We now must write out each of the nineteen expressions above and rewrite the entire expression in $p' - q'$ -exponential form. The first term

$$\begin{aligned}
\sum_{j=1}^{n+1} \mathcal{S}_j(q) \frac{\partial r}{\partial q'_j}(p', q') &= \sum_{j=1}^{n+1} \left[\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} s_k(p') e^{ik \cdot q} \right) \right] = \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_j s_k(p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \mathcal{S}_{m,j} k_j s_k(p') e^{im \cdot q} \right) e^{ik \cdot q} \right] = \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{S}_{m,j} k_j s_k(p') e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{S}_{m,j} k_j s_{w-m}(p') \right) e^{iw \cdot q}.
\end{aligned}$$

The second term

$$\begin{aligned}
\sum_{j=1}^{n+1} \mathcal{S}_j(q) \frac{\partial g}{\partial q'_j}(p', q') &= \sum_{j=1}^{n+1} \left[\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q} \right) \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,j} e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_j h_k(p') e_k(t) e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,n+1} e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} h_k(p') \frac{de_k}{dt}(t) e^{ik \cdot q}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \mathcal{S}_{m,j} k_j h_k(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{S}_{m,n+1} h_k(p') \frac{de_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{S}_{m,j} k_j h_k(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{S}_{m,n+1} h_k(p') \frac{de_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{S}_{m,j} k_j h_{w-m}(p') e_{w-m}(t) \right) e^{iw \cdot q} + \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{S}_{m,n+1} h_{w-m}(p') \frac{de_{w-m}}{dt}(t) \right) e^{iw \cdot q}.
\end{aligned}$$

The third term

$$\begin{aligned}
\sum_{j=1}^{n+1} \mathcal{F}_j(q') \frac{\partial r}{\partial q'_j}(p', q') &= \sum_{j=1}^{n+1} \left[\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} s_k(p') e^{ik \cdot q} \right) \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_j s_k(p') e^{ik \cdot q} \right] = \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \mathcal{F}_{m,j}(t) k_j s_k(p') e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{F}_{m,j}(t) k_j s_k(p') e^{im \cdot q} \right) e^{ik \cdot q} = \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{F}_{m,j}(t) (w-m)_j s_{w-m}(p') \right) e^{iw \cdot q}.
\end{aligned}$$

The fourth term

$$\begin{aligned}
\sum_{j=1}^{n+1} \mathcal{F}_j(q') \frac{\partial g}{\partial q'_j}(p', q') &= \sum_{j=1}^{n+1} \left[\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} h_k(p') e_k(t) e^{ik \cdot q} \right) \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,j}(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} i k_j h_k(p') e_k(t) e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,n+1}(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} h_k(p') \frac{de_k}{dt}(t) e^{ik \cdot q} \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} i \mathcal{F}_{m,j}(t) k_j h_k(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{F}_{m,n+1}(t) h_k(p') \frac{de_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{F}_{m,j}(t) k_j h_k(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{F}_{m,n+1}(t) h_k(p') \frac{de_k}{dt}(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n i \mathcal{F}_{m,j}(t) (w-m)_j h_{w-m}(p') e_{w-m}(t) \right) e^{iw \cdot q} + \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \mathcal{F}_{m,n+1}(t) h_{w-m}(p') \frac{de_{w-m}}{dt}(t) \right) e^{iw \cdot q}.
\end{aligned}$$

The fifth term

$$\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{Y}}{\partial q'_j}(q) \frac{\partial f}{\partial p'_j}(p') \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q} \right) \frac{\partial f}{\partial p'_j}(p') \right] = \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} i k_j y_k e^{ik \cdot q} \frac{\partial f}{\partial p'_j}(p') \right] = \sum_{k \in \mathbb{Z}^n} \left(\sum_{j=1}^n i k_j y_k \frac{\partial f}{\partial p'_j}(p') \right) e^{ik \cdot q}.$$

The sixth term

$$\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{Y}}{\partial q'_j}(q) \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q} \right) \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right]$$

$$\begin{aligned}
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} ik_j y_k e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] = \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j y_m \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j y_m \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) e^{ik \cdot q} = \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j y_m \frac{\partial s_{w-m}}{\partial p'_j}(p') \right) e^{iw \cdot q}.
\end{aligned}$$

The seventh term

$$\begin{aligned}
&\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{Y}}{\partial q'_j}(q) \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} y_k e^{ik \cdot q} \right) \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} ik_j y_k e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] = \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j y_m \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j y_m \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} = \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j y_m \frac{\partial h_{w-m}}{\partial p'_j}(p') e_{w-m}(t) \right) e^{iw \cdot q}.
\end{aligned}$$

The eighth term

$$\begin{aligned}
&\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{T}}{\partial q'_j}(q') \frac{\partial f}{\partial p'_j}(p') \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \right) \frac{\partial f}{\partial p'_j}(p') \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} ik_j x_k(t) e^{ik \cdot q} \frac{\partial f}{\partial p'_j}(p') \right] + \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} \frac{\partial f}{\partial p'_{n+1}}(p') \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{j=1}^n ik_j x_k(t) \frac{\partial f}{\partial p'_j}(p') \right) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \left(\frac{dx_k}{dt}(t) \frac{\partial f}{\partial p'_{n+1}}(p') \right) e^{ik \cdot q}.
\end{aligned}$$

The ninth term

$$\begin{aligned}
&\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{T}}{\partial q'_j}(q') \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \right) \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} ik_j x_k(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{ik \cdot q} \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j x_m(t) \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) e^{ik \cdot q} \right] + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j x_m(t) \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) e^{ik \cdot q} + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j x_m(t) \frac{\partial s_{w-m}}{\partial p'_j}(p') \right) e^{iw \cdot q} + \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial s_{w-m}}{\partial p'_{n+1}}(p') \right) e^{iw \cdot q}.
\end{aligned}$$

The tenth term

$$\sum_{j=1}^{n+1} \left[\frac{\partial \mathcal{T}}{\partial q'_j}(q') \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] = \sum_{j=1}^{n+1} \left[\frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} x_k(t) e^{ik \cdot q} \right) \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right]$$

$$\begin{aligned}
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} ik_j x_k(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] \\
&+ \sum_{k \in \mathbb{Z}^n} \frac{dx_k}{dt}(t) e^{ik \cdot q} \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{ik \cdot q} \\
&= \sum_{j=1}^n \left[\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j x_m(t) \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&+ \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j x_m(t) \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&+ \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{j=1}^n im_j x_m(t) \frac{\partial h_{w-m}}{\partial p'_j}(p') e_{w-m}(t) \right) e^{iw \cdot q} \\
&+ \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{dx_m}{dt}(t) \frac{\partial h_{w-m}}{\partial p'_{n+1}}(p') e_{w-m}(t) \right) e^{iw \cdot q}.
\end{aligned}$$

The eleventh term

$$\sum_{j=1}^{n+1} \left[\xi_j \frac{\partial f}{\partial p'_j}(p') \right].$$

The twelve-th term

$$\sum_{j=1}^{n+1} \left[\xi_j \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] = \sum_{k \in \mathbb{Z}^n} \left(\sum_{j=1}^{n+1} \xi_j \frac{\partial s_k}{\partial p'_j}(p') \right) e^{ik \cdot q}.$$

The thirteenth term

$$\sum_{j=1}^{n+1} \left[\xi_j \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] = \sum_{k \in \mathbb{Z}^n} \left(\sum_{j=1}^{n+1} \xi_j \frac{\partial h_k}{\partial p'_j}(p') e_k(t) \right) e^{ik \cdot q}.$$

The fourteenth term

$$\begin{aligned}
\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j}(q) p'_l \frac{\partial f}{\partial p'_j}(p') \right] &= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,l} e^{ik \cdot q} \right) p'_l \frac{\partial f}{\partial p'_j}(p') \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{S}_{k,l} e^{ik \cdot q} p'_l \frac{\partial f}{\partial p'_j}(p') \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left(\sum_{j=1}^n \sum_{l=1}^{n+1} ik_j \mathcal{S}_{k,l} p'_l \frac{\partial f}{\partial p'_j}(p') \right) e^{ik \cdot q}.
\end{aligned}$$

The fifteenth term

$$\begin{aligned}
\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j} (q) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j} (p') e^{ik \cdot q} \right] &= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,l} e^{ik \cdot q} \right) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j} (p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{S}_{k,l} e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j} (p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial s_k}{\partial p'_j} (p') e^{im \cdot q} \right) e^{ik \cdot q} \right) \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial s_k}{\partial p'_j} (p') e^{im \cdot q} \right) \right] e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial s_{w-m}}{\partial p'_j} (p') \right) \right] e^{iw \cdot q}.
\end{aligned}$$

The sixteenth term

$$\begin{aligned}
\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{S}_l}{\partial q'_j} (q) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j} (p') e_k(t) e^{ik \cdot q} \right] &= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{S}_{k,l} e^{ik \cdot q} \right) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j} (p') e_k(t) e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{S}_{k,l} e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j} (p') e_k(t) e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial h_k}{\partial p'_j} (p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right) \right] \\
&= \sum_{k \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial h_k}{\partial p'_j} (p') e_k(t) e^{im \cdot q} \right) \right] e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{S}_{m,l} p'_l \frac{\partial h_{w-m}}{\partial p'_j} (p') e_{w-m}(t) \right) \right] e^{iw \cdot q}.
\end{aligned}$$

The seventeenth term

$$\begin{aligned}
\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j} (q') p'_l \frac{\partial f}{\partial p'_j} (p') \right] &= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,l}(t) e^{ik \cdot q} \right) p'_l \frac{\partial f}{\partial p'_j} (p') \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{F}_{k,l}(t) e^{ik \cdot q} p'_l \frac{\partial f}{\partial p'_j} (p') \right] \\
&\quad + \sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \frac{d\mathcal{F}_{k,l}}{dt}(t) e^{ik \cdot q} p'_l \frac{\partial f}{\partial p'_{n+1}} (p')
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} ik_j p'_l \frac{\partial f}{\partial p'_j}(p') \mathcal{F}_{k,l}(t) \right] e^{ik \cdot q} \\
&+ \sum_{k \in \mathbb{Z}^n} \left(\sum_{l=1}^{n+1} p'_l \frac{\partial f}{\partial p'_{n+1}}(p') \frac{d\mathcal{F}_{k,l}}{dt}(t) \right) e^{ik \cdot q}.
\end{aligned}$$

The eighteenth term

$$\begin{aligned}
&\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,l}(t) e^{ik \cdot q} \right) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{F}_{k,l}(t) e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \right] \\
&+ \sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \frac{d\mathcal{F}_{k,l}}{dt}(t) e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{ik \cdot q} \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
&+ \sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{k \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial s_k}{\partial p'_j}(p') e^{im \cdot q} \right) \right] e^{ik \cdot q} \\
&+ \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{l=1}^{n+1} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial s_k}{\partial p'_{n+1}}(p') e^{im \cdot q} \right) e^{ik \cdot q} \\
&= \sum_{w \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial s_{w-m}}{\partial p'_j}(p') \right) \right] e^{iw \cdot q} \\
&+ \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{l=1}^{n+1} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial s_{w-m}}{\partial p'_{n+1}}(p') \right) e^{iw \cdot q}.
\end{aligned}$$

The nineteenth term

$$\begin{aligned}
&\sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial \mathcal{F}_l}{\partial q'_j}(q') p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] \\
&= \sum_{j=1}^{n+1} \left[\sum_{l=1}^{n+1} \frac{\partial}{\partial q'_j} \left(\sum_{k \in \mathbb{Z}^n} \mathcal{F}_{k,l}(t) e^{ik \cdot q} \right) p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right] \\
&= \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} ik_j \mathcal{F}_{k,l}(t) e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \frac{d\mathcal{F}_{k,l}}{dt}(t) e^{ik \cdot q} p'_l \sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{ik \cdot q} \\
& = \sum_{j=1}^n \left[\sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \right] \\
& + \sum_{l=1}^{n+1} \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
& = \sum_{k \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{im \cdot q} \right) \right] e^{ik \cdot q} \\
& + \sum_{k \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{l=1}^{n+1} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial h_k}{\partial p'_{n+1}}(p') e_k(t) e^{im \cdot q} \right) e^{ik \cdot q} \\
& = \sum_{w \in \mathbb{Z}^n} \left[\sum_{j=1}^n \sum_{l=1}^{n+1} \left(\sum_{m \in \mathbb{Z}^n} im_j \mathcal{F}_{m,l}(t) p'_l \frac{\partial h_{w-m}}{\partial p'_j}(p') e_{w-m}(t) \right) \right] e^{iw \cdot q} \\
& + \sum_{w \in \mathbb{Z}^n} \left(\sum_{m \in \mathbb{Z}^n} \sum_{l=1}^{n+1} \frac{d\mathcal{F}_{m,l}}{dt}(t) p'_l \frac{\partial h_{w-m}}{\partial p'_{n+1}}(p') e_{w-m}(t) \right) e^{iw \cdot q}.
\end{aligned}$$

All the nineteen terms have one of three forms. Terms one, five, six, twelve, fourteen, and fifteen have the form

$$\sum_{k \in \mathbb{Z}^n} f_k(p') e^{ik \cdot q}.$$

Terms two, three, four, seven, eight, nine, ten, thirteen, sixteen, seventeen, eighteen and nineteen have the form

$$\sum_{k \in \mathbb{Z}^n} g_k(p') h_k(t) e^{ik \cdot q}.$$

Term eleven is simply a function of p' , say $L(p')$. Adding all terms together we see the final result, $\{\chi, F\}$, is of $p' - q'$ -exponential-form. Finally we have the following analyticity domains

$$\begin{aligned}
Y_j(q') & \in \mathcal{A}_{\rho', \sigma'}, \quad \frac{\partial r}{\partial q'_j}(p', q') \in \mathcal{A}_{\rho-\delta, \sigma}, \quad \frac{\partial g}{\partial q'_j}(p', q') \in \mathcal{A}_{\rho, \sigma-\delta}, \quad \frac{\partial X}{\partial q'_j}(q') \in \mathcal{A}_{\rho', \sigma'-\delta}, \\
\sum_{l=1}^{n+1} \frac{\partial Y_l}{\partial q'_j}(q') p'_l & \in \mathcal{A}_{\rho', \sigma'-\delta}, \quad \frac{\partial f}{\partial p'_j}(p') \in \mathcal{A}_{\rho, \sigma}, \quad \sum_{k \in \mathbb{Z}^n} \frac{\partial s_k}{\partial p'_j}(p') e^{ik \cdot q} \in \mathcal{A}_{\rho, \sigma}, \\
\sum_{k \in \mathbb{Z}^n} \frac{\partial h_k}{\partial p'_j}(p') e_k(t) e^{ik \cdot q} & \in \mathcal{A}_{\rho, \sigma}
\end{aligned}$$

for the functions that make up the expression for $\{\chi, F\}(p', q')$. Since these functions are multiplied together in a number of ways and then summed it follows, given $\rho' < \rho$ and $\sigma' < \sigma$, $\{\chi, F\}(p', q') \in \mathcal{A}_{\rho'-\delta, \sigma'-\delta}$ and is of $p' - q'$ -exponential form. \square

Lemma F.8

Let $H(p', q') = U(p', q') + P(p', q')$ with H and P being of p', q' -exponential form. Define $\mathcal{R}_A = r_2(U, \chi, 1) + r_1(P, \chi, 1)$, where

$$r_m(H, \chi, t) = \mathcal{U}H - \sum_{l=0}^{m-1} \frac{t^l}{l!} L_\chi^l H = \sum_{l=m}^{\infty} \frac{t^l}{l!} L_\chi^l H,$$

$L_\chi^0 H = H$ and $L_\chi^m H = \{L_\chi^{m-1} H, H\}$ for $m \geq 1$. \mathcal{R}_A is of p', q' -exponential form.

Proof:

We have

$$\mathcal{R}_A = \{\chi, P\} + \mathcal{U}H - H - \{\chi, H\} = r_2(U, \chi, 1) + r_1(P, \chi, 1),$$

where

$$r_m(H, \chi, t) = \mathcal{U}H - \sum_{l=0}^{m-1} \frac{t^l}{l!} L_\chi^l H = \sum_{l=m}^{\infty} \frac{t^l}{l!} L_\chi^l H.$$

We thus have

$$\mathcal{R}_A = \{\chi, P\} + r_2(H, \chi, 1).$$

By lemma F.7 with $\chi \in \mathcal{A}_{\rho', \sigma'}$ and for some positive $\delta < \sigma'$ it follows $\{\chi, P\} \in \mathcal{A}_{\rho' - \delta, \sigma' - \delta}$ is of $p' - q'$ -exponential form. Next we examine

$$r_2(H, \chi, 1) = \sum_{l=2}^{\infty} \frac{1}{l!} L_\chi^l H.$$

Clearly by inductively applying lemma F.7 $r_2(H, \chi, 1)$ is of $p' - q'$ -exponential form in some domain, $\mathcal{D}_{\rho^*, \sigma^*}$, to be determine. We set

$$\rho^* = \rho - \sum_{i=0}^{\infty} \delta_i,$$

$$\sigma^* = \sigma - \sum_{i=0}^{\infty} \delta_i.$$

Each time we apply lemma F.7, we can choose δ_i arbitrarily small. In particular we can set $\delta_i = \delta_{i-1}/2$ for $i = 1, 2, \dots$ with $\delta_0 = \delta$. Therefore

$$\rho^* = \rho - \sum_{i=0}^{\infty} \delta \left(\frac{1}{2}\right)^i = \rho - 2\delta,$$

$$\sigma^* = \sigma - \sum_{i=0}^{\infty} \delta \left(\frac{1}{2}\right)^i = \sigma - 2\delta.$$

We finally have $\mathcal{R}_A \in \mathcal{A}_{\rho^*, \sigma^*}$ is of $p' - q'$ -exponential form. \square

G Rossby Wave Flow

The Rossby wave flow is generated by a Hamiltonian of the form

$$\begin{aligned} H(X, Y, t) &= H^0(X, Y, t) + \varepsilon H^1(X, Y, t) \\ &= A \sin k_0(X - c_0 t) \sin l_0 Y + \varepsilon H^1(X, Y, t), \end{aligned}$$

where $\varepsilon > 0$, A is the maximum velocity in the y -direction, (k_0, l_0) is the wave number vector, and c_0 is the phase speed of the primary wave in the x -direction. In a reference frame moving with the primary wave the transformation

$$x = X - c_0 t, \quad y = Y,$$

yields the Hamiltonian

$$\begin{aligned} H(x, y, t) &= H^0(x, y, t) + \varepsilon H^1(x, y, t) \\ &= -c_0 y + A \sin k_0 x \sin l_0 y + \varepsilon H^1(x, y, t). \end{aligned}$$

In this frame of reference the vector field generated by the Hamiltonian $H(x, y, t)$ has the following form:

$$\begin{aligned} \dot{x} &= c_0 - A l_0 \sin k_0 x \cos l_0 y - \varepsilon \frac{\partial H^1}{\partial y}(x, y, t), \\ \dot{y} &= A k_0 \cos k_0 x \sin l_0 y + \varepsilon \frac{\partial H^1}{\partial x}(x, y, t). \end{aligned}$$

We will consider the perturbation $H^1(x, y, t) = xy \operatorname{sech}^2 at$ where a is the decay rate.

The near integrable form of this problem makes it a candidate to apply the theorem presented in this paper provided the conditions of the hypothesis are satisfied. The first step in applying the theorem is to transform the Hamiltonian to action-angle variables. Clearly, by the Liouville-Arnold Theorem [4] one can construct locally for the one degree of freedom system generated by the Hamiltonian $H^0(x, y)$ a symplectic coordinate transformation to action-angle variables. The integrable system is characterized by two heteroclinic connections between saddle-type equilibrium points on the curves $y = 0$, $y = \pi$ respectively and by the symmetry trajectory starting at $(0, \pi/2)$ shown in Figure 1. These three structures divide the phase space in four regions. Action-angle variables can be found locally for each of these four regions as we will now describe. The action variable p is defined as

$$p = \frac{1}{2\pi} \int_H x dy, \tag{G.1}$$

which is the area of the region inside the level set with Hamiltonian H divided by 2π . Figure 2 shows the actions for the system in Figure 1 starting from the right at $p = 0$ for the elliptic equilibrium point in region I and rising to a maximum at $H = -\pi/4$ for the symmetry trajectory. Similarly for regions III and IV the action falls from a maximum at the symmetry trajectory to zero for the equilibrium point in region IV . Note $I(H)$ is invertible in each of the intervals corresponding to regions I, II, III and IV . To define the angle $\theta(x, y)$, let L denote a straight curve emanating from the elliptic equilibrium point in region I to the elliptic equilibrium point in region IV . We denote solutions of the integrable vector field starting on L by $(x(t, s), y(t, s))$ where $x(0, s) = x_0(s)$ and $y(0, s) = y_0(s)$ so for any point (x, y) on the orbit $(x(t, s), y(t, s))$, $t = t(x, y)$ is the time it takes for the solution starting at $(x_0(s), y_0(s))$ to reach (x, y) . Given $T(H)$ is the period of the periodic orbit with constant H , we define the angle variable, $\theta(x, y)$ as

$$\theta(x, y) = 2\pi \frac{t(x, y)}{T(H)}, \tag{G.2}$$

where $(x, y) \in H = \text{constant}$. Clearly by this definition action-angle variables can not be defined on the heteroclinic connections.

Next we show the Hamiltonian in action-angle variables is analytic.

Lemma G.1 *The transformation to action-angle variables is analytic.*

Proof G.1 *We begin by transforming the integral part of the Hamiltonian $H^0(x, y)$ in region I . In region I (G.1) has the following form*

$$p(H^0) = \frac{1}{\pi} \int_{y_{\min}}^{y_{\max}} x(y, H^0) dy = \frac{1}{\pi} \int_{y_{\min}}^{y_{\max}} \arcsin \left[\frac{y + H^0}{\sin y} - \frac{\pi}{2} \right] dy, \tag{G.3}$$

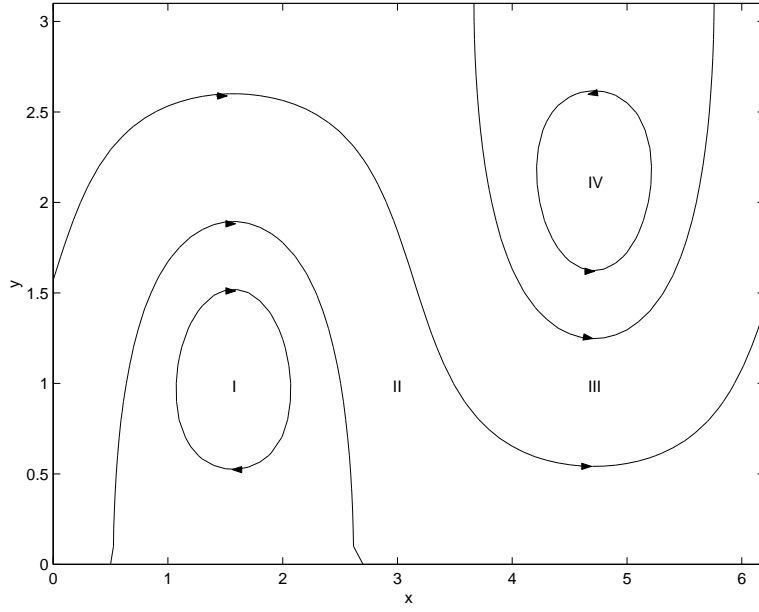


Figure 1: Phase space of the vector field generated by the Hamiltonian $H = -c_0 y + A \sin k_0 x \sin l_0 y$ with $A = k_0 = l_0 = 1.0$ and $c_0 = 0.5$.

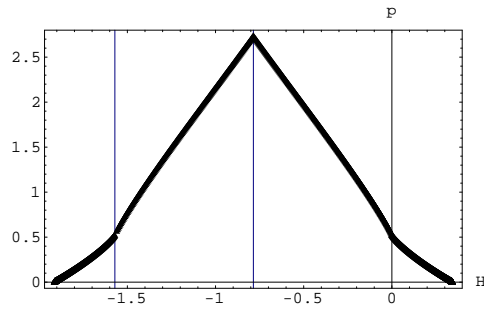


Figure 2: Action map for each of the four regions starting from the right I, II, III, IV .

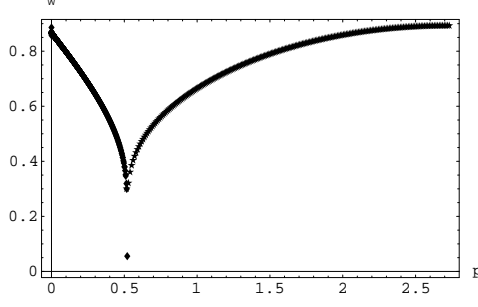


Figure 3: Frequency Map.

where the $\pi/2$ term translates the integral to the y -axis and y_{\max} , y_{\min} are the values of y where the level set intersects the y axis. We consider a complex extension of the variables H^0 , $H_C = H^0 + iH_{Im}$, and y , $y_C = y + iy_{Im}$, to complex strips and for sake of simplicity do not specify their width. The integral in (G.3) becomes a contour integral along the real axis with end points y_{\max} , y_{\min} . Let $\mathcal{Y} = (Y_{\min}, Y_{\max})$ denote the interval of y values in region I. Since $T(y, H^0)$ and $T_{H^0}(y, H^0)$ are continuous for H_C in a complex extension, D , of $H^0(\mathcal{Y})$ and $T(y, H^0)$ and $T_{H^0}(y, H^0)$ are continuous for y_C on the contour consisting of the real interval (y_{\max}, y_{\min}) , it follows $p(H^0)$ is analytic in D [10]. Furthermore, by the inverse function theorem, except for the elliptic equilibria where $p = 0$, $H^0(p)$ is analytic on some strip of p . Similarly, the proof for regions II, III, IV follows.

For the angle variable in region I (G.2) becomes

$$\theta(x, y) = \frac{2\pi}{T(H)} \int_{y_{\min}}^y \frac{dy}{\sqrt{\sin^2 y - (H^0 + \frac{y}{2})^2}}, \quad T(H) = 2 \int_{y_{\min}}^{y_{\max}} \frac{dy}{\sqrt{\sin^2 y - (H^0 + \frac{y}{2})^2}}.$$

By the same arguments as for the action transformation and defining a complex extension of x , $x_C = x + ix_{Im}$, it follows $\theta(x, y)$ is analytic for some strip of x_C and y_C .

When transforming the perturbation $H^1(x, y, t) = xy \operatorname{sech}^2 at$ to the action-angle variables found for H^0 , it is clear that the time dependent term $\operatorname{sech}^2 at$ is the same after the transformation. By the Liouville-Arnold Theorem the action-angle transformation $(x, y) \rightarrow (p(H^0), \theta(x, y))$ is invertible, $(p, \theta) \rightarrow (x(p, \theta), y(p, \theta))$, and by the inverse function theorem $x(p, \theta)$ and $y(p, \theta)$ are analytic. \square

Note, since the time dependence of the perturbation is $\operatorname{sech}^2 at$, the perturbation is of exponential order with respect to time as required by the hypothesis of the theorem.

Finally the nondegeneracy condition given by $\det(\partial^2 H^0 / \partial p^2) = \det(\partial \omega / \partial p) \neq 0$ must be satisfied. We numerically compute the period and plot the frequency as a function of the action giving the frequency map shown in Figure 3. This graph gives the frequencies for the closed orbits in regions I, II as well as the closed orbits in regions III, IV. The frequency at $p = 0$ corresponds to both elliptic equilibrium points in regions I and IV. The frequency falls to zero for a p value corresponding to both heteroclinic connections as one would expect. The frequencies to the right of this value of p correspond to the closed orbits in regions II and III reaching a maximum for the symmetry trajectory. The nondegeneracy condition is therefore satisfied for all tori except for the symmetry trajectory.

H Aside Calculations

We take an aside to check (9.2) reduces to the appropriate expression if the time dependence of $g_k(s)$ is not aperiodic but is indeed periodic with frequency ω_0 . Without loss of generality we assume ω_0 to be positive. In this case we write the Fourier series expansion of $g_k(s)$

$$g_k(s) = \sum_{l \in \mathbb{Z}} g_{kl} e^{il\omega_0 s}$$

and substitute this in (9.2) which results in

$$f_k(t) = f_k(0)e^{-i(\tilde{\lambda} \cdot k)t} + \int_0^t \sum_{l \in \mathbb{Z}} g_{kl} e^{il\omega_0 s} e^{i(\tilde{\lambda} \cdot k)(s-t)} ds.$$

Now let $\sum_{l \in \mathbb{Z}} s_l = 1$ be any series whose sum is one. Then we have

$$\begin{aligned} f_k(t) &= f_k(0)e^{-i(\tilde{\lambda} \cdot k)t} + \sum_{l \in \mathbb{Z}} g_{kl} \int_0^t e^{il\omega_0 s} e^{i(\tilde{\lambda} \cdot k)(s-t)} ds \\ &= f_k(0)e^{-i(\tilde{\lambda} \cdot k)t} \sum_{l \in \mathbb{Z}} s_l + \sum_{l \in \mathbb{Z}} g_{kl} \int_0^t e^{il\omega_0 s} e^{i(\tilde{\lambda} \cdot k)(s-t)} ds \\ &= \sum_{l \in \mathbb{Z}} f_k(0)e^{-i(\tilde{\lambda} \cdot k)t} s_l + \sum_{l \in \mathbb{Z}} g_{kl} \int_0^t e^{il\omega_0 s} e^{i(\tilde{\lambda} \cdot k)(s-t)} ds \\ &= \sum_{l \in \mathbb{Z}} \left[f_k(0)s_l e^{-i(\tilde{\lambda} \cdot k)t} + \frac{g_{kl} e^{-i(\tilde{\lambda} \cdot k)t}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \left(e^{i(\tilde{\lambda} \cdot k + \omega_0 l)t} - 1 \right) \right] \\ &= \sum_{l \in \mathbb{Z}} \left[f_k(0)s_l e^{-i(\tilde{\lambda} \cdot k + \omega_0 l)t} + \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \left(1 - e^{-i(\tilde{\lambda} \cdot k + \omega_0 l)t} \right) \right] e^{i\omega_0 l t} \\ &= \sum_{l \in \mathbb{Z}} \left[\left(f_k(0)s_l - \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \right) e^{-i(\tilde{\lambda} \cdot k + \omega_0 l)t} + \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \right] e^{i\omega_0 l t} \end{aligned} \quad (H.1)$$

Note, that except for the time dependence inside the square bracket, (H.1) is almost in the form of the Fourier series for $f_k(t)$

$$f_k(t) = \sum_{l \in \mathbb{Z}} f_{kl} e^{il\omega_0 t}. \quad (H.2)$$

Comparing (H.1) and (H.2) we see that we can eliminate the time dependence in the square bracket by setting

$$f_k(0)s_l = \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)}.$$

By summing on both sides of the equation, this is equivalent to

$$\begin{aligned} \sum_{l \in \mathbb{Z}} f_k(0)s_l &= \sum_{l \in \mathbb{Z}} \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \\ f_k(0) \sum_{l \in \mathbb{Z}} s_l &= \sum_{l \in \mathbb{Z}} \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \\ f_k(0) &= \sum_{l \in \mathbb{Z}} \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} \end{aligned} \quad (H.3)$$

In which case, (H.1) reduces to

$$f_k(t) = \sum_{l \in \mathbb{Z}} \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)} e^{il\omega_0 t} \quad (H.4)$$

with

$$f_{kl} = \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + \omega_0 l)}.$$

Note that (H.4) reduces to the expression for $f_k(0)$ obtained in (H.3).

We now must obtain an estimate for $|f_k(t)|$. Since $G(q, t), F(q, t) \in \mathcal{A}_\rho$, the Fourier coefficients $f_k(t)$ and $g_k(t)$ must be exponentially small with respect to the index k . First we analyze the estimate for $g_k(t)$.

We have

$$\begin{aligned} G(q, t) &= \sum_{q \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q} \\ &= \sum_{q \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}} g_{kl} e^{il\omega_0 t} e^{ik \cdot q}. \end{aligned}$$

We calculate $|g_{kl}|$

$$g_{kl} = \frac{1}{(2\pi)^n} \frac{1}{S} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{-S/2}^{S/2} G(q, s) e^{-ik \cdot q} e^{-i\omega_0 l s} ds dq$$

where the integral is along the real axis. Lift the integral as follows

$$\begin{aligned} g_{kl} &= \frac{1}{(2\pi)^n} \frac{1}{S} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{-S/2}^{S/2} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho, s - i \frac{l}{|l|} \sigma) \\ &\quad \cdot \left\{ \prod_{j=1}^n e^{-ik_j (q_j - i \frac{k_j}{|k_j|} \rho)} \right\} e^{-il\omega_0 (s - i \frac{l}{|l|} \sigma)} dq ds \\ g_{kl} &= \frac{e^{-|k|\rho} e^{-|l|\omega_0 \sigma}}{(2\pi)^n S} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{-S/2}^{S/2} G(q_1 - i \frac{k_1}{|k_1|} \rho, \dots, q_n - i \frac{k_n}{|k_n|} \rho, s - i \frac{l}{|l|} \sigma) \\ &\quad \cdot e^{-ik \cdot q} e^{-i\omega_0 l s} ds dq \end{aligned}$$

and

$$|g_{kl}| \leq e^{-|k|\rho} e^{-|l|\omega_0 \sigma} \|G(q, t)\|_{\rho, \sigma}.$$

Now, since

$$g_k(t) = \sum_{l \in \mathbb{Z}} g_{kl} e^{il\omega_0 t},$$

to calculate the estimate $|g_k(t)|$ we need to calculate the following

$$|e^{il\omega_0 t}| = \left(e^{il\omega_0 t} e^{-il\omega_0 \bar{t}} \right)^{1/2}$$

where $t \in \mathbb{C}$ and \bar{t} indicates the complex conjugate. Then

$$|e^{il\omega_0 t}| = e^{-\omega_0 l(\operatorname{Im} t)}$$

where $\operatorname{Im} t$ is the imaginary part of t . Assume $|\operatorname{Im} t| \leq \sigma - \delta$. Then

$$-\omega_0 l(\operatorname{Im} t) \leq |\omega_0 l(\operatorname{Im} t)| \leq |l| |\omega_0| (\sigma - \delta).$$

We then obtain

$$\begin{aligned} |g_k(t)| &\leq \sum_{l \in \mathbb{Z}} |g_{kl}| |e^{il\omega_0 t}| \\ &\leq e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \sum_{l \in \mathbb{Z}} e^{-|l|\omega_0 \sigma} e^{|l|\omega_0(\sigma - \delta)} \\ &\leq e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \sum_{l \in \mathbb{Z}} e^{-|l|\omega_0 \delta} \\ &\leq 2e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \sum_{l \in \mathbb{Z}^+} e^{-\omega_0 \delta l} \\ &= 2e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \left(\frac{1}{1 - e^{-\omega_0 \delta}} \right). \end{aligned}$$

We proceed similarly to obtain an estimate for $f_k(t)$. We know

$$f_k(t) = \sum_{l \in \mathbb{Z}} \frac{g_{kl} e^{il\omega_0 t}}{i(\tilde{\lambda} \cdot k + l\omega_0)}.$$

We obtain from the previous estimates

$$|f_k(t)| \leq e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \sum_{l \in \mathbb{Z}} \frac{e^{-|l|\omega_0 \delta}}{|\tilde{\lambda} \cdot k + l\omega_0|}.$$

The diophantine condition on the frequencies gives an estimate for the denominator of this expression. First recall that for $k \in \mathbb{Z}^n$ we define $|k| = \sum_{i=1}^n |k_i|$ and $\|k\| = \sup_i |k_i|$. Furthermore, define the vectors $h = (k, l)$ and $\Delta = (\tilde{\lambda}, \omega_0)$. Consider first the case $h = (k, l) \neq 0$. The diophantine condition is given by

$$|\Delta \cdot h| \geq \Gamma \|h\|^{-(n+1)}.$$

For any $K, Y, \delta' > 0$ recall the inequality

$$K^Y \leq \left(\frac{Y}{e\delta'} \right)^Y e^{K\delta'}.$$

It follows that

$$\begin{aligned}
\frac{1}{|\Delta \cdot h|} &\leq \Gamma^{-1} \|h\|^{n+1} \\
&\leq \Gamma^{-1} |h|^{n+1} \\
&\leq \Gamma^{-1} \left(\frac{n+1}{e\delta'} \right)^{n+1} e^{|h|\delta'}.
\end{aligned}$$

We choose

$$\delta' = \delta \frac{\omega_0}{\omega_0 + 1} < \delta$$

so that

$$\begin{aligned}
|f_k(t)| &\leq e^{-|k|\rho} \|G(q, t)\|_{\rho, \sigma} \Gamma^{-1} \left(\frac{(n+1)(\omega_0 + 1)}{e\omega_0\delta} \right)^{n+1} \sum_{l \in \mathbb{Z}} e^{-|l|\omega_0\delta} e^{|h|\frac{\delta\omega_0}{\omega_0+1}} \\
&= \|G(q, t)\|_{\rho, \sigma} \Gamma^{-1} \left(\frac{(n+1)(\omega_0 + 1)}{e\omega_0\delta} \right)^{n+1} e^{-|k|(\rho - \frac{\delta\omega_0}{\omega_0+1})} \sum_{l \in \mathbb{Z}} e^{-|l|\omega_0\delta \left(\frac{\omega_0}{\omega_0+1} \right)} \\
&\leq \|G(q, t)\|_{\rho, \sigma} \Gamma^{-1} \left(\frac{(n+1)(\omega_0 + 1)}{e\omega_0\delta} \right)^{n+1} e^{-|k|(\rho - \delta)} \sum_{l \in \mathbb{Z}} e^{-|l|\omega_0\delta} \\
&\leq \|G(q, t)\|_{\rho, \sigma} \Gamma^{-1} \left(\frac{(n+1)(\omega_0 + 1)}{e\omega_0\delta} \right)^{n+1} \left(\frac{2}{1 - e^{-\omega_0\delta}} \right) e^{-|k|(\rho - \delta)}
\end{aligned}$$

For the case $h = (k, l) = 0$ we use the assumption $\overline{G}(q, t) = 0$, where $\overline{G}(q, t)$ represents the average of the function over the variables q and t . In this case, we use the following Lemma to show $f_0(t)$ or f_{00} are bounded.

Lemma H.1

Given $G(q, t)$ is periodic with respect to q and t and has Fourier coefficients g_{kl} , $g_{00} = 0$ iff $\overline{G}(q, t) = 0$.

Proof:

We have

$$G(q, t) = \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}} g_{kl} e^{i\omega_0 l t} e^{ik \cdot q}$$

and

$$\begin{aligned}
\overline{G}(q, t) &= \left(\frac{1}{2\pi} \right)^n \frac{1}{S} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \int_{-s/2}^{s/2} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}} g_{kl} e^{i\omega_0 l t} e^{ik \cdot q} dt dq \\
&= \left(\frac{1}{2\pi} \right)^n \frac{1}{S} \sum_{k \in \mathbb{Z}^n} \sum_{l \in \mathbb{Z}} \left[g_{kl} \left[\frac{e^{i\omega_0 l t}}{i\omega_0 l} \right]_{-s/2}^{s/2} \left[\frac{e^{ik \cdot q}}{(i)^n \prod_{j=1}^n k_j} \right]_{-\pi}^{\pi} \right] \\
&= g_{00}
\end{aligned}$$

□

Therefore since

$$f_{kl} = \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + l\omega_0)}$$

we have $f_{00} = 0$ and given

$$f_k(t) = \sum_{l \in \mathbb{Z}} \frac{g_{kl} e^{il\omega_0 t}}{i(\tilde{\lambda} \cdot k + l\omega_0)}$$

it follows

$$\begin{aligned}
|f_0(t)| &\leq \sum_{\substack{l \in \mathbb{Z} \\ l \neq 0}} \frac{|g_{0l}| |e^{il\omega_0 t}|}{|l\omega_0|} \\
&\leq \|G(q, t)\|_{\rho, \sigma} \gamma^{-1} \left(\frac{(n+1)(\omega_0+1)}{e\omega_0\delta} \right)^{n+1} \left(\frac{2}{1-e^{-\omega_0\delta}} \right)
\end{aligned}$$

Now we want to check that, given $G(q, t)$ is periodic with respect to time, (9.10) reduces to the appropriate expression for $f_k(t)$, (H.4). Assuming $G(q, t)$ is time periodic with frequency ω_0 implies the Fourier coefficients $f_k(t)$ are periodic functions with

$$G(q, t) = \sum_{k \in \mathbb{Z}^n} g_k(t) e^{ik \cdot q}$$

and

$$g_k(t) = \sum_{l \in \mathbb{Z}} g_{kl} e^{il\omega_0 t}.$$

The Fourier transform of a periodic function is equals to a series of the following from

$$\mathcal{G}_k(\omega) = \sum_{l \in \mathbb{Z}} g_{kl} \delta(\omega - l\omega_0). \quad (\text{H.5})$$

That is, the Fourier transform consists of equally spaced delta functions which are weighted according to the Fourier coefficients of the function.

Substituting (H.5) in (9.10) gives the following

$$\begin{aligned}
f_k(t) &= \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\mathcal{G}_k(\omega)}{i(\tilde{\lambda} \cdot k + \omega)} e^{i\omega t} d\omega \\
&= \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\sum_{l \in \mathbb{Z}} g_{kl} \delta(\omega - l\omega_0)}{i(\tilde{\lambda} \cdot k + \omega)} e^{i\omega t} d\omega \\
&= \sum_{l \in \mathbb{Z}} g_{kl} \int_{-\infty+i\beta}^{\infty+i\beta} \frac{\delta(\omega - l\omega_0)}{i(\tilde{\lambda} \cdot k + \omega)} e^{i\omega t} d\omega. \quad (\text{H.6})
\end{aligned}$$

We use the following property of the delta function for a function $h(\omega)$ continuous at $\omega = 0$

$$\int_{-\infty+i\beta}^{\infty+i\beta} \delta(\omega) h(\omega) d\omega = h(0)$$

and (H.6) becomes

$$f_k(t) = \sum_{l \in \mathbb{Z}} \frac{g_{kl}}{i(\tilde{\lambda} \cdot k + l\omega_0)} e^{il\omega_0 t}.$$

We derive a bound for the sum $\sum_{m \in \mathbb{Z}^n} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)}$. We examine the cases $n = 1$, $n = 2$ and derive an expression for the general case.

$$\boxed{n = 1}$$

We obtain two bounds, one for $k \geq 0$ and another for $k < 0$. These bounds are equal and thus the sum $\sum_{m \in \mathbb{Z}^n} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)}$ is bounded for any k .

Let $k \geq 0$ and divide the range of m in three intervals, $m \leq 0$, $0 < m < k$ and $m \geq k$. We obtain bounds for the three intervals

$$\begin{aligned} m \leq 0 \quad & \left\{ \sum_{m \leq 0} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq e^{-|k|(\rho-2\delta)} \sum_{m \leq 0} e^{-|m|\rho}, \right. \\ 0 < m < k \quad & \left\{ \begin{aligned} \sum_{0 < m < k} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} &= \sum_{0 < m < k} e^{-|k|(\rho-2\delta)} e^{-|m|(2\delta)} \\ &\leq e^{-|k|(\rho-2\delta)} \sum_{0 < m} e^{-|m|(2\delta)}, \end{aligned} \right. \\ m \geq k \quad & \left\{ \sum_{m \geq k} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq e^{-|k|\rho} \sum_{k-m \leq 0} e^{-|k-m|(\rho-2\delta)}. \right. \end{aligned}$$

Therefore for $k \geq 0$

$$\begin{aligned} \sum_{m \in \mathbb{Z}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} &\leq e^{-|k|(\rho-2\delta)} \sum_{m \leq 0} e^{-|m|\rho} \\ &+ e^{-|k|(\rho-2\delta)} \sum_{0 < m} e^{-|m|(2\delta)} \\ &+ e^{-|k|\rho} \sum_{k-m \leq 0} e^{-|k-m|(\rho-2\delta)} \\ &\leq 3e^{-|k|(\rho-2\delta)} \sum_{m \geq 0} e^{-|m|(2\delta)}. \end{aligned}$$

Let $k < 0$ and divide the range of m in three intervals, $m \leq k$, $k < m < 0$ and $m \geq 0$. We obtain bounds for the three intervals.

$$\begin{aligned} m \leq k \quad & \left\{ \sum_{m \leq k} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq e^{-|k|\rho} \sum_{k-m \geq 0} e^{-|k-m|(\rho-2\delta)}, \right. \\ k < m < 0 \quad & \left\{ \begin{aligned} \sum_{k < m < 0} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} &= \sum_{k < m < 0} e^{k(\rho-2\delta)} e^{m(2\delta)} \\ &= e^{-|k|(\rho-2\delta)} \sum_{k < m < 0} e^{-|m|(2\delta)} \\ &\leq e^{-|k|(\rho-2\delta)} \sum_{m < 0} e^{-|m|(2\delta)}, \end{aligned} \right. \end{aligned}$$

$$m \geq 0 \quad \left\{ \quad \sum_{m \geq 0} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} = e^{-|k|(\rho-2\delta)} \sum_{m \geq 0} e^{-|m|\rho} \right.$$

Therefore for $k < 0$

$$\begin{aligned} \sum_{m \in \mathbb{Z}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} &\leq e^{-|k|\rho} \sum_{k-m \geq 0} e^{-|k-m|(\rho-2\delta)} \\ &+ e^{-|k|(\rho-2\delta)} \sum_{m < 0} e^{-|m|(2\delta)} \\ &+ e^{-|k|(\rho-2\delta)} \sum_{m \geq 0} e^{-|m|\rho} \\ &\leq 3e^{-|k|(\rho-2\delta)} \sum_{m \geq 0} e^{-|m|(2\delta)}. \end{aligned}$$

Finally for any $k \in \mathbb{Z}$ we have the bound

$$\sum_{m \in \mathbb{Z}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq 3e^{-|k|(\rho-2\delta)} \sum_{m \geq 0} e^{-|m|(2\delta)} = 3e^{-|k|(\rho-2\delta)} \left(\frac{1}{1 - e^{-2\delta}} \right).$$

$$\boxed{n = 2}$$

In this case we have vectors $k = (k_1, k_2) \in \mathbb{Z}^2$ and $m = (m_1, m_2) \in \mathbb{Z}^2$. We must keep in mind four sub-cases (1) $k_1 \geq 0$ and $k_2 \geq 0$, (2) $k_1 \geq 0$ and $k_2 < 0$, (3) $k_1 < 0$ and $k_2 \geq 0$, (3) $k_1 < 0$ and $k_2 < 0$. We will find a bound for sub-case (1) and the rest will follow without loss of generality. We now find the bound for $k_1, k_2 \geq 0$.

$$\begin{aligned} \begin{matrix} m_1 \leq 0 \\ m_2 \leq 0 \end{matrix} \quad \left\{ \quad \sum_{\substack{m_1 \leq 0 \\ m_2 \leq 0}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \leq 0 \\ m_2 \leq 0}} e^{-|m|\rho}, \right. \\ \\ \begin{matrix} m_1 \leq 0 \\ 0 < m_2 < k_2 \end{matrix} \quad \left\{ \quad \sum_{\substack{m_1 \leq 0 \\ 0 < m_2 < k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \\ \\ \leq \sum_{\substack{m_1 \leq 0 \\ 0 < m_2 < k_2}} e^{-|m_1|\rho} e^{-|k_1|(\rho-2\delta)} e^{-|k_2|(\rho-2\delta)} e^{-|m_2|(2\delta)} \\ \\ = e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \leq 0 \\ 0 < m_2 < k_2}} e^{-|m_1|\rho} e^{-|m_2|(2\delta)} \\ \\ \leq e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 \\ 0 < m_2}} e^{-|m|(2\delta)}, \\ \\ \begin{matrix} m_1 \leq 0 \\ m_2 > k_2 \end{matrix} \quad \left\{ \quad \sum_{\substack{m_1 \leq 0 \\ m_2 > k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\substack{m_1 \leq 0 \\ m_2 > k_2}} e^{-|m_1|\rho} e^{-|k_1|(\rho-2\delta)} e^{-|k_2|\rho} e^{-|k_2-m_2|(\rho-2\delta)} \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \leq 0 \\ m_2 > k_2}} e^{-|m_1|\rho} e^{-|k_2-m_2|(\rho-2\delta)},
\end{aligned}$$

$$\begin{aligned}
\begin{matrix} 0 < m_1 < k_1 \\ m_2 \leq 0 \end{matrix} &\left\{ \sum_{\substack{0 < m_1 < k_1 \\ m_2 \leq 0}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 < k_1 \\ m_2 \leq 0}} e^{-|m_2|\rho} e^{-|m_1|(2\delta)} \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{m_2 \leq 0} e^{-|m_2|\rho},
\end{aligned}$$

$$\begin{aligned}
\begin{matrix} 0 < m_1 < k_1 \\ 0 < m_2 < k_2 \end{matrix} &\left\{ \sum_{\substack{0 < m_1 < k_1 \\ 0 < m_2 < k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \\
&\leq \sum_{\substack{0 < m_1 < k_1 \\ 0 < m_2 < k_2}} e^{-|k_1|(\rho-2\delta)} e^{-|m_1|(2\delta)} e^{-|k_2|(\rho-2\delta)} e^{-|m_2|(2\delta)} \\
&= e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 < k_1 \\ 0 < m_2 < k_2}} e^{-|m_1|(2\delta)} e^{-|m_2|(2\delta)} \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 \\ 0 < m_2}} e^{-|m_1|(2\delta)} e^{-|m_2|(2\delta)},
\end{aligned}$$

$$\begin{aligned}
\begin{matrix} 0 < m_1 < k_1 \\ m_2 \geq k_2 \end{matrix} &\left\{ \sum_{\substack{0 < m_1 < k_1 \\ m_2 \geq k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \\
&\leq \sum_{\substack{0 < m_1 < k_1 \\ m_2 \geq k_2}} e^{-|k_1|(\rho-2\delta)} e^{-|m_1|(2\delta)} e^{-|k_2|\rho} e^{-|k_2-m_2|(\rho-2\delta)} \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 < k_1 \\ m_2 \geq k_2}} e^{-|m_1|(2\delta)} e^{-|k_2-m_2|(\rho-2\delta)},
\end{aligned}$$

$$\begin{aligned}
\begin{matrix} m_1 \geq k_1 \\ m_2 \leq 0 \end{matrix} &\left\{ \sum_{\substack{m_1 \geq k_1 \\ m_2 \leq 0}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \right. \\
&\leq \sum_{\substack{m_1 \geq k_1 \\ m_2 \leq 0}} e^{-|k_1|\rho} e^{-|m_2|\rho} e^{-|k_1-m_1|(\rho-2\delta)} e^{-|k_2|(\rho-2\delta)} \\
&\leq e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \geq k_1 \\ m_2 \leq 0}} e^{-|m_2|\rho} e^{-|k_1-m_1|(\rho-2\delta)},
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{array}{l} m_1 \geq k_1 \\ 0 < m_2 < k_2 \end{array} \right\} \sum_{\substack{m_1 \geq k_1 \\ 0 < m_2 < k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \\
& \leq \sum_{\substack{m_1 \geq k_1 \\ 0 < m_2 < k_2}} e^{-|k_1|\rho} e^{-|k_2|(\rho-2\delta)} e^{-|m_2|(2\delta)} e^{-|k_1-m_1|(\rho-2\delta)} \\
& \leq e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \geq k_1 \\ 0 < m_2 < k_2}} e^{-|m_2|(2\delta)} e^{-|k_1-m_1|(\rho-2\delta)},
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{array}{l} m_1 \geq k_1 \\ m_2 \geq k_2 \end{array} \right\} \sum_{\substack{m_1 \geq k_1 \\ m_2 \geq k_2}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \\
& \leq \sum_{\substack{m_1 \geq k_1 \\ m_2 \geq k_2}} e^{-|k_1|\rho} e^{-|k_1-m_1|(\rho-2\delta)} e^{-|k_2|\rho} e^{-|k_2-m_2|(\rho-2\delta)} \\
& \leq e^{-|k|\rho} \sum_{\substack{m_1 \geq k_1 \\ m_2 \geq k_2}} e^{-|k-m|(\rho-2\delta)}.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\sum_{\substack{m_1 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} & \leq e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \leq 0 \\ m_2 \leq 0}} e^{-|m|\rho} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 \\ 0 < m_2}} e^{-|m|(2\delta)} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \leq 0 \\ m_2 > k_2}} e^{-|m_1|\rho} e^{-|k_2-m_2|(\rho-2\delta)} \\
& + e^{-|k|(\rho-2\delta)} \sum_{m_2 \leq 0} e^{-|m_2|\rho} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 \\ 0 < m_2}} e^{-|m_1|(2\delta)} e^{-|m_2|(2\delta)} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{0 < m_1 < k_1 \\ m_2 \geq k_2}} e^{-|m_1|(2\delta)} e^{-|k_2-m_2|(\rho-2\delta)} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \geq k_1 \\ m_2 \leq 0}} e^{-|m_2|\rho} e^{-|k_1-m_1|(\rho-2\delta)} \\
& + e^{-|k|(\rho-2\delta)} \sum_{\substack{m_1 \geq k_1 \\ 0 < m_2 < k_2}} e^{-|m_2|(2\delta)} e^{-|k_1-m_1|(\rho-2\delta)} \\
& + e^{-|k|\rho} \sum_{\substack{m_1 \geq k_1 \\ m_2 \geq k_2}} e^{-|k-m|(\rho-2\delta)} \\
& \leq 9e^{-|k|(\rho-2\delta)} \sum_{\substack{0 \geq m_1 \\ 0 \geq m_2}} e^{-|m_1|(2\delta)} e^{-|m_2|(2\delta)}
\end{aligned}$$

$$= 9e^{-|k|(\rho-2\delta)} \left(\frac{1}{1-e^{-2\delta}} \right)^2.$$

Without loss of generality the same bound is obtained for the cases involving different values of k_1 and k_2 . Therefore we have for any $k \in \mathbb{Z}^2$

$$\sum_{m \in \mathbb{Z}^2} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq 9e^{-|k|(\rho-2\delta)} \left(\frac{1}{1-e^{-2\delta}} \right)^2.$$

Following the same procedure we can write a bound for the arbitrary n case. That is, for any $k \in \mathbb{Z}^n$

$$\sum_{m \in \mathbb{Z}^n} e^{-|m|\rho} e^{-|k-m|(\rho-2\delta)} \leq 3^n e^{-|k|(\rho-2\delta)} \left(\frac{1}{1-e^{-2\delta}} \right)^n.$$

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